The supersingular locus of the Shimura variety of GU$(1, n-1)$ II

Inken Vollaard, Torsten Wedhorn

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Abstract. We complete the study of the supersingular locus $\mathcal{M}^{\mathrm{ss}}$ in the fiber at p of a Shimura variety attached to a unitary similitude group $\mathrm{GU}(1, n-1)$ over $\mathbb{Q}$ in the case that p is inert. This was started by the first author in [Vo] where complete results were obtained for $n = 2, 3$. The supersingular locus $\mathcal{M}^{\mathrm{ss}}$ is uniformized by a formal scheme $\mathcal{N}$ which is a moduli space of so-called unitary p-divisible groups. It depends on the choice of a unitary isocrystal $\mathbf{N}$. We define a stratification of $\mathcal{N}$ indexed by vertices of the Bruhat-Tits building attached to the reductive group of automorphisms of $\mathbf{N}$. We show that the combinatorial behaviour of this stratification is given by the simplicial structure of the building. The closures of the strata (and in particular the irreducible components of $\mathcal{N}_{\mathrm{red}}$) are identified with (generalized) Deligne-Lusztig varieties. We show that the Bruhat-Tits stratification is a refinement of the Ekedahl-Oort stratification and also relate the Ekedahl-Oort strata to Deligne-Lusztig varieties. We deduce that $\mathcal{M}^{\mathrm{ss}}$ is locally a complete intersection, that its irreducible components and each Ekedahl-Oort stratum in every irreducible component is isomorphic to a Deligne-Lusztig variety, and give formulas for the number of irreducible components of every Ekedahl-Oort stratum of $\mathcal{M}^{\mathrm{ss}}$.

Introduction

Let (G, X) be a Shimura datum, where G is a reductive group over $\mathbb{Q}$ such that $G_{\mathbb{R}}$ is isomorphic to the group of unitary similitudes of a hermitian space of signature $(1, n-1)$ for some integer $n > 1$. The reflex field of (G, X) is contained in a quadratic imaginary extension E of $\mathbb{Q}$. We fix a prime $p > 2$ such that p is *inert* in E and we assume that G is unramified at p. Let $C^p \subset G(\mathbb{A}_f^p)$ be an open compact subgroup and let $\mathcal{M}_{C^p}$ be the associated moduli space of abelian varieties defined by Kottwitz [Ko] for these PEL-Shimura data. It is defined over $\mathbb{Z}_{p^2} := \mathcal{O}_{E_v}$ where v is the unique place of E lying over p. We refer to Section 5 for the precise definition.

In this paper we complete the study of the supersingular locus $\mathcal{M}^{\mathrm{ss}}_{C^p}$ of the reduction modulo p of $\mathcal{M}_{C^p}$, which was started in [Vo]. The uniformization theorem of Rapoport and Zink ([RZ] Theorem 6.30) provides us with an isomorphism (see (5.4))

$$\text{(U)} \qquad \coprod_j \Gamma_j \backslash \mathcal{N}_{\mathrm{red}} \xrightarrow{\sim} \mathcal{M}^{\mathrm{ss}}_{C^p}$$

where $\mathcal{N}_{\mathrm{red}}$ is the underlying reduced subscheme of a moduli space $\mathcal{N}$ of quasi-isogenies of unitary p-divisible groups (X, ρ_X) (see (1.5) for its definition) and where $(\Gamma_j)_j$ is a finite family of discrete groups acting continuously on $\mathcal{N}$. The moduli space $\mathcal{N}$ is represented by a separated formal scheme locally formally of finite type over $\mathrm{Spf}(\mathbb{Z}_{p^2})$. We study the structure of $\mathcal{N}$ and of $\mathcal{N}_{\mathrm{red}}$.

By [Vo], we have a decomposition $\mathcal{N} = \coprod_{i\in\mathbb{Z}} \mathcal{N}_i$ where $\mathcal{N}_i$ classifies p-isogenies of height ni. Moreover, $\mathcal{N}_i$ is nonempty if and only if ni is even and in this case $\mathcal{N}_i$ is isomorphic to $\mathcal{N}_0$. For the remainder of the introduction let i be fixed such that ni is even.

We define a stratification of $\mathcal{N}_{i,\mathrm{red}}$ by locally closed subschemes $\mathcal{N}^0_\Lambda$, where Λ runs through the set of vertices of a Bruhat-Tits building $\mathcal{B}$ associated with the data above. More precisely, denote by $\mathbf{N}$ a superspecial unitary isocrystal over $\mathbb{F}_{p^2}$ (see (1.4)). The automorphisms of $\mathbf{N}$ form an algebraic group J over $\mathbb{Q}_p$ which is an inner form of $G_{\mathbb{Q}_p}$. Let $\tilde{J}$ be the derived group of J, denote by $\mathcal{B} = \mathcal{B}(\tilde{J}, \mathbb{Q}_p)$ its simplicial Bruhat Tits building, and let $\mathcal{B}_0$ be the set of vertices of $\mathcal{B}$. Every such vertex can be considered as a lattice in a nondegenerate $(K/\mathbb{Q}_p))$-hermitian space, where K is a quadratic unramified extension of $\mathbb{Q}_p$. For each such lattice $\Lambda \in \mathcal{B}_0$ we define a projective subscheme $\mathcal{N}_\Lambda$ of $\mathcal{N}_{i,\mathrm{red}}$ (see (3.3) for the definition).

In [Vo] subsets $\mathcal{V}(\Lambda)(k)$ of $\mathcal{N}_i(k)$ were defined for each algebraically closed extension k of $\mathbb{F}_{p^2}$ and it was conjectured that these subsets are the sets of a k-valued points of a subscheme of $\mathcal{N}_i$. This conjecture was proved in loc. cit. for $n = 2$ and $n = 3$. As $\mathcal{N}_\Lambda(k) = \mathcal{V}(\Lambda)(k)$, the construction of $\mathcal{N}_\Lambda$ proves this conjecture for arbitrary n.

To describe the structure of $\mathcal{N}_\Lambda$ we introduce the following notation. For each $\Lambda \in \mathcal{B}_0$ let $\tilde{J}_\Lambda$ be the maximal reductive quotient of the special fibre of the parahoric group scheme of $\tilde{J}$ associated with the vertex Λ. Then $\tilde{J}_\Lambda$ is the special unitary group of an $(\mathbb{F}_{p^2}/\mathbb{F}_p)$-hermitian space V_Λ of odd dimension, say $2t(\Lambda)+1$. The group $\tilde{J}(\mathbb{Q}_p)$ acts on $\mathcal{B}_0$ and the map $\Lambda \mapsto t(\Lambda)$ induces a bijection between the set of $\tilde{J}(\mathbb{Q}_p)$-orbits on $\mathcal{B}_0$ and the set of integers $\{0, 1, \dots, [(n-1)/2]\}$. We call $t(\Lambda)$ the orbit type of $\Lambda \in \mathcal{B}_0$.

The schemes $\mathcal{N}_\Lambda$ have the following properties (see Corollary 3.11).

Theorem A. *$\mathcal{N}_\Lambda$ is projective, smooth, and geometrically irreducible of dimension $t(\Lambda)$. If $t(\Lambda) = t(\Lambda')$, the $\mathbb{F}_{p^2}$-schemes $\mathcal{N}_\Lambda$ and $\mathcal{N}_{\Lambda'}$ are isomorphic.*

For the proof of Theorem A we construct an isomorphism of $\mathcal{N}_\Lambda$ with a Deligne-Lusztig variety Y_Λ of the group $\tilde{J}_\Lambda$ (see (3.5)). Here we mean by a Deligne-Lusztig variety of an algebraic group H (defined over a finite field) a variety classifying parabolic subgroups P of H of a fixed conjugacy type such that P and $F(P)$ are in a fixed relative position, where F is the Frobenius. This generalizes the varieties considered by Deligne and Lusztig who studied the case where the conjugacy class of parabolic subgroups is the class of Borel subgroups. In (3.4) we collect some results about these (generalized) Deligne-Lusztig varieties.

The intersection behaviour of the subschemes $\mathcal{N}_\Lambda$ is given by the following result (see Theorem 4.1 and Proposition 4.3).

Theorem B. *Let $\Lambda, \Lambda' \in \mathcal{B}_0$.*
(1) *The Bruhat-Tits stratum $\mathcal{N}_{\Lambda'}$ is contained in $\mathcal{N}_\Lambda$ if and only if $\Lambda' \subset \Lambda$.*
(2) *Two Bruhat-Tits strata $\mathcal{N}_\Lambda$ and $\mathcal{N}_{\Lambda'}$ have a nonempty interscetion if and only if $\Lambda \cap \Lambda' \in \mathcal{B}_0$, and in this case $\mathcal{N}_\Lambda \cap \mathcal{N}_{\Lambda'} = \mathcal{N}_{\Lambda\cap\Lambda'}$ scheme-theoretically.*
(3) *Let $\mathcal{N}_\Lambda^0$ be the open subscheme of $\mathcal{N}_\Lambda$ which is the complement of the union of the (finitely many) closed subschemes $\mathcal{N}_{\Lambda'}$ with $\mathcal{N}_{\Lambda'} \subsetneq \mathcal{N}_\Lambda$. Then the natural morphism*

$$\coprod_{\Lambda\in\mathcal{B}_0} \mathcal{N}_\Lambda^0 \longrightarrow \mathcal{N}_i$$

is bijective.
(4) *The subscheme $\mathcal{N}_\Lambda^0$ is open and dense in $\mathcal{N}_\Lambda$.*

In particular, we obtain the stratification of $\mathcal{N}_i$ by the $\mathcal{N}_\Lambda^0$ mentioned above. Theorem B shows that the intersection behaviour and the inclusion relation of the subschemes $\mathcal{N}_\Lambda$ are given by the combinatorial structure of the Bruhat-Tits building of $\tilde{J}$. Therefore we call this stratification the *Bruhat-Tits stratification*. As an application we show that any kind of intersection behaviour within the bounds given by Theorem B can occur (Proposition 4.5) and we give formulas how many closed Bruhat-Tits strata of a fixed dimension are contained in and are containing a given one (Corollary 4.7).

Moreover, we obtain the following corollary (Theorem 4.2 and Theorem 4.11).

Corollary C. *$\mathcal{N}_i$ is geometrically connected of pure dimension $[(n-1)/2]$ and locally a complete intersection. The irreducible components of $\mathcal{N}_{i,\mathrm{red}}$ are precisely the projective subschemes $\mathcal{N}_\Lambda$ for $\Lambda \in \mathcal{B}_0$ with $t(\Lambda) = [(n-1)/2]$.*

Then we compare the Bruhat-Tits stratification with the Ekedahl-Oort stratification of $\mathcal{N}_i$ given by the isomorphism class of the p-torsion of the

unitary p-divisible group (defined in (2.3)). We show that the Ekedahl-Oort strata are smooth (Theorem 2.2) and that the former stratification is a refinement of the latter one (see (4.3)):

Theorem D. *Let $x, x' \in \mathcal{N}_0(k)$, where k is an algebraically closed extension of $\mathbb{F}_{p^2}$. Then these points lie in the same Ekedahl-Oort stratum if and only $x \in \mathcal{N}^0_\Lambda(k)$ and $x' \in \mathcal{N}^0_{\Lambda'}(k)$ with $t(\Lambda) = t(\Lambda')$.*

We obtain the following corollary (see Corollary 4.12 and Theorem 4.11).

Corollary E. *The smooth locus of $\mathcal{N}_i$ is the open Ekedahl-Oort stratum of $\mathcal{N}_i$. Each Ekedahl-Oort stratum of $\mathcal{N}_\Lambda$ is isomorphic to a Deligne-Lusztig variety.*

Using the theory of p-adic uniformization mentioned above we obtain now the following result for the local structure of the supersingular locus $\mathcal{M}^{\mathrm{ss}}_{C^p}$ (see Theorem 5.2).

Theorem F. *The supersingular locus $\mathcal{M}^{\mathrm{ss}}_{C^p}$ is of pure dimension $[(n-1)/2]$ and locally of complete intersection. Its smooth locus is the open Ekedahl-Oort stratum $\mathcal{M}^{\mathrm{ss}}_{C^p}([(n-1)/2])$.*

Note that by [Wd] all Ekedahl-Oort strata of $\mathcal{M}_{C^p}$ are smooth. We also obtain formulas for the number of connected components of $\mathcal{M}^{\mathrm{ss}}_{C^p}$ and the number of irreducible components of all Ekedahl-Oort strata (and in particular for the number of irreducible components for $\mathcal{M}^{\mathrm{ss}}_{C^p}$) (see Proposition 5.3 and Proposition 5.4). These formulas in particular show that within a connected component of $\mathcal{M}^{\mathrm{ss}}_{C^p}$ the number of irreducible components of a given Ekedahl-Oort stratum becomes arbitrarily large when p goes to infinity.

If C^p is sufficiently small, the irreducible components of $\mathcal{M}^{\mathrm{ss}}_{C^p}$ are isomorphic to the irreducible components of $\mathcal{N}_i$ (which are all pairwise isomorphic by Theorem A). We illustrate the above results in the following low-dimensional example (the case $n = 3$ has already been studied in [Vo], for $n = 4$ see (4.2) and Proposition 5.5).

Example G. *Let $n = 3$ or $n = 4$ and let C^p be sufficiently small. The supersingular locus $\mathcal{M}^{\mathrm{ss}}_{C^p}$ is projective, equi-dimensional of dimension 1, and locally a complete intersection. Each irreducible component is isomorphic to the Fermat curve in $\mathbb{P}^2$ given by the equation $x_0^{p+1} + x_1^{p+1} + x_2^{p+1} = 0$. Its singular points are the superspecial points of $\mathcal{M}^{\mathrm{ss}}_{C^p}$ and there are $p^3 + 1$ of them on any irreducible component. Each superspecial point is the pairwise transversal intersection of $p + 1$ (for $n = 3$), resp. of $p^3 + 1$ (for $n = 4$) irreducible components.*

The geometric results obtained in this paper are used by S. Kudla and M. Rapoport in [KR] where they study the local intersection theory of special cycles in the unramified case.

S. Harashita has obtained in [Ha] a description in terms of Deligne-Lusztig varieties for the union of certain Ekedahl-Oort strata for the moduli space of principally polarized abelian varieties. In fact, some results of loc. cit. can be reformulated using the language of Bruhat-Tits buildings and this reformulation initiated in part this paper. Harashita's results have been refined by M. Hoeve in [Ho] using so-called fine Deligne-Lusztig varieties. We remark that this is analogous to the results described here, as the Deligne-Lusztig varieties of Corollary 4.12 can be also described as fine Deligne-Lusztig varieties.

The paper is organized as follows. The first section fixes notations and defines the formal scheme $\mathcal{N}$. Section 2 studies the Ekedahl Oort stratification of $\mathcal{N}$. In Section 3 the schemes $\mathcal{N}_\Lambda$ are defined and it is proved that they are isomorphic to certain Deligne-Lusztig varieties. This is used in Section 4 to prove the main results on the structure of $\mathcal{N}$. In Section 5 we explain how these results are applied to the structure of the supersingular locus of Shimura varieties attached to certain unitary groups.

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1 The Moduli space $\mathscr{N}$ of p-divisible groups

(1.1) The local PEL-datum.

Let $p > 2$ be a fixed prime number. Let K be an unramified extension of $\mathbb{Q}_p$ of degree 2 and denote by * the nontrivial Galois automorphism of K over $\mathbb{Q}_p$. Let V be a finite-dimensional K-vector space and set $n = \dim_K(V)$. Let $\langle\,,\rangle$ be a $\mathbb{Q}_p$-valued skew-hermitian form on V, i.e., $\langle\,,\rangle\colon V \times V \to \mathbb{Q}_p$ is an alternating $\mathbb{Q}_p$-bilinear form such that $\langle av, w\rangle = \langle v, a^*w\rangle$. Let G be the algebraic group over $\mathbb{Q}_p$ such that

$$G(R) = \{\, g \in \mathrm{GL}_{K\otimes R}(V_R) \mid \exists c \in R^\times\colon \langle gv, gw\rangle = c\langle v, w\rangle \forall v, w \in V \otimes R \,\}$$

for every $\mathbb{Q}_p$-algebra R. This is a reductive group over $\mathbb{Q}_p$.

There exists a unique skew-hermitian form $(\ ,\ )'\colon V \times V \to K$ such that $\langle v, w\rangle = \mathrm{Tr}_{K/\mathbb{Q}_p}(v, w)'$. Moreover, if $\delta \in K^\times$ with $\delta^* = -\delta$, $(v, w) := \delta(v, w)'$ is a hermitian form, and G is the reductive group of similitudes of the hermitian space $(V, (\ ,\ ))$.

Let O_K be the ring of integers of K. We assume that there exists an O_K-lattice Γ such that $\langle\,,\rangle$ induces a perfect $\mathbb{Z}_p$-pairing on Γ. This implies that G has a reductive model over $\mathbb{Z}_p$, namely the group of O_K-linear symplectic similitudes of $(\Gamma, \langle\,,\rangle_{|\Gamma\times\Gamma})$.

(1.2) Unitary p-divisible groups.

Let $\mathbb{F}_{p^2}$ be a finite field with p^2 elements and define $\mathbb{Z}_{p^2} := W(\mathbb{F}_{p^2})$ and let $\mathbb{Q}_{p^2}$ be its field of fractions. We denote by φ_0 and $\varphi_1 = \varphi_0 \circ {}^*$ the two $\mathbb{Q}_p$-isomorphisms of K to $\mathbb{Q}_{p^2}$. Let (Nilp) be the category of $\mathbb{Z}_{p^2}$-schemes S, such that p is locally nilpotent on S.

We fix an integer r with $0 \le r \le n$. For each $S \in$ (Nilp) a *p-divisible O_K-module of signature $(r, n-r)$ over S* is a pair $X = (X, \iota_X)$, where X is a p-divisible group over S with an O_K-action $\iota_X\colon O_K \to \operatorname{End}(X)$ satisfying

$$\text{(1.2.1)} \qquad \operatorname{charpol}(\iota(a)|\operatorname{Lie}(X)) = (T - \varphi_0(a))^r (T - \varphi_1(a))^{n-r} \in \mathbb{Z}_{p^2}[T]$$

for all $a \in \mathcal{O}_K$. We call this the *signature condition* $(r, n-r)$. Note that (1.2.1) implies $\operatorname{rk}_{\mathscr{O}_S}(\operatorname{Lie}(X)) = n$.

We denote by ${}^tX$ the dual p-divisible group endowed with the O_K-action $\iota_{{}^tX}(a) = {}^t(\iota_X(a^*))$. A *unitary p-divisible group of signature* $(r, n-r)$ is a triple $X = (X, \iota_X, \lambda_X)$ where (X, ι_X) is a p-divisible O_K-module of signature $(r, n-r)$ and where $\lambda_X\colon X \to {}^tX$ is a p-principal O_K-linear polarization. The existence of λ_X implies

$$\operatorname{height}(X) = \operatorname{rk}_{\mathscr{O}_S}(\operatorname{Lie}(X)) + \operatorname{rk}_{\mathscr{O}_S}(\operatorname{Lie}({}^tX)) = 2n.$$

We call two unitary p-divisible groups X and Y *isomorphic* if there exists an $\mathcal{O}_K$-linear isomorphism $\alpha\colon X \to Y$ such that ${}^t\alpha \circ \lambda_Y \circ \alpha$ is a $\mathbb{Z}_p^\times$-multiple of λ_X.

Finally, by repeating the definitions above for at level 1 truncated Barsotti-Tate groups we obtain the notion of a *unitary* BT_1 *of signature* $(r, n-r)$.

(1.3) Dieudonné modules of unitary p-divisible groups.

Now assume that $S = \operatorname{Spec}(k)$ where k is a perfect field extension of $\mathbb{F}_{p^2}$. Then the covariant Dieudonné module of a unitary p-divisible groups of signature $(r, n-r)$ is a Dieudonné module $(M, \mathcal{F}, \mathcal{V})$ of height $2n$ over k endowed with an $\mathcal{O}_K$-action and a perfect alternating $W(k)$-bilinear pairing $\langle\,,\rangle$ on M satisfying

$$\text{(1.3.1)} \qquad \langle \mathcal{F}x, y\rangle = \langle x, \mathcal{V}y\rangle^\sigma, \qquad \langle ax, y\rangle = \langle x, a^*y\rangle$$

for all $x, y \in M$ and $a \in \mathcal{O}_K$. Here σ denotes the Frobenius on $W(k)$.

Via the decomposition

$$(1.3.2) \qquad \mathcal{O}_K \otimes_{\mathbb{Z}_p} W(k) \xrightarrow{\sim} W(k) \times W(k), \quad a \otimes w \mapsto (\varphi_0(a)w, \varphi_1(a)w),$$

the $\mathcal{O}_K$-action on M defines a decomposition $M = M_0 \oplus M_1$ and the conditions (1.3.1) mean that $\mathcal{F}$ and $\mathcal{V}$ are homogeneous of degree 1 with respect to this decomposition and that M_0 and M_1 are totally isotropic with respect to $\langle\,,\rangle$. Finally the signature condition (1.2.1) is equivalent to the equalities

$$\dim_k(M_0/\mathcal{V}M_1) = r, \qquad \dim_k(M_1/\mathcal{V}M_0) = n-r.$$

We call such tuples $M = (M, \mathcal{F}, \mathcal{V}, M = M_0 \oplus M_1, \langle\,,\rangle)$ *unitary Dieudonné modules of signature* $(r, n-r)$.

If $M = (M, \mathcal{F}, \mathcal{V}, M = M_0 \oplus M_1, \langle\,,\rangle)$ is a unitary Dieudonné module of signature $(r, n-r)$ over k we can consider its reduction $\bar{M} = M \otimes_{W(k)} k$ which is a $\mathbb{Z}/2\mathbb{Z}$-graded k-vector space of dimension $2n$ endowed with $\mathcal{F}$ and $\mathcal{V}$ of degree 1 and a perfect alternating pairing $\langle\,,\rangle$ such that $\bar{M}_0$ and $\bar{M}_1$ are totally isotropic with respect to $\langle\,,\rangle$ and such that $\dim_k(\bar{M}_0/\mathcal{V}\bar{M}_1) = r$ and $\dim_k(\bar{M}_1/\mathcal{V}\bar{M}_0) = n-r$. We call such a tuple

$$\bar{M} = (\bar{M}, \mathcal{F}, \mathcal{V}, \bar{M} = \bar{M}_0 \oplus \bar{M}_1, \langle\,,\rangle)$$

a *unitary Dieudonné space of signature* $(r, n-r)$.

(1.4) The standard unitary p-divisible group $\mathbf{X}$.

We fix a unitary p-divisible group $\mathbf{X} = (\mathbf{X}, \iota, \lambda)$ over $\mathbb{F}_{p^2}$ of signature $(r, n-r)$. We let $(\mathbf{M}, \mathbf{F}, \mathbf{V})$ be the covariant Dieudonné module of $\mathbf{X}$. Then $\mathbf{M}$ is a free $\mathbb{Z}_{p^2}$-module of rank $2n$. Via ι it is endowed with an O_K-action and λ defines a perfect symplectic pairing on $\mathbf{M}$. Therefore $\mathbf{M}$ is a unitary Dieudonné module over $\mathbb{F}_{p^2}$.

We denote by $(\mathbf{N}, \mathbf{F}) = (\mathbf{M}, \mathbf{F}) \otimes \mathbb{Q}_{p^2}$ the isocrystal of $\mathbf{X}$.

We assume that $\mathbf{X}$ is superspecial and that the isocrystal $(\mathbf{N}, \mathbf{F})$ of $\mathbf{X}$ is generated by elements n such that $\mathbf{F}^2 n = pn$. In particular, $\mathbf{N}$ is decent in the sense of [RZ] 2.13. As $\mathbf{F}^2$ is a $\mathbb{Q}_{p^2}$-linear map, we have $\mathbf{F}^2 = p\,\mathrm{id}_{\mathbf{N}}$ and therefore $\mathbf{F} = \mathbf{V}$.

Such a triple $(\mathbf{X}, \iota, \lambda)$ exists. Indeed, we define its unitary Dieudonné module $\mathbf{M}$ as follows. Fix $\delta \in \mathbb{Z}_{p^2}^{\times}$ such that $\sigma(\delta) = -\delta$. We consider $\tilde{SS} = O_K \otimes \mathbb{Z}_{p^2} = \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^2}$ and set $g := (1,0), h := (0,1) \in \tilde{SS}$. Define a $(\mathbb{Z}_{p^2}, \sigma)$-linear map $\mathbf{F}$ on $\tilde{SS}$ by $\mathbf{F}(h) = g$ and $\mathbf{F}(g) = ph$ and define a $(\mathbb{Z}_{p^2}, \sigma^{-1})$-linear map $\mathbf{V}$ on $\tilde{SS}$ by $\mathbf{V}(h) = g$ and $\mathbf{V}(g) = ph$. We define a perfect $\mathbb{Z}_{p^2}$-pairing on $\tilde{SS}$ by $\langle g, h\rangle = \delta$. This makes $\tilde{SS}$ into a unitary Dieudonné module of signature $(0,1)$ and we set

$$\mathbf{M} = \tilde{SS}^{n-r} \oplus \sigma^*(\tilde{SS})^r,$$

where $\sigma^*(\tilde{SS})$ is $\tilde{SS}$ as an $O_K \otimes \mathbb{Z}_{p^2}$-module with symplectic pairing but where for the definition of $\mathbf{F}$ and $\mathbf{V}$ the roles of g and h are interchanged.

(1.5) The moduli space of p-divisible groups.

For each scheme $S \in$ (Nilp) we set $\bar{S} = S \otimes_{\mathbb{Z}_{p^2}} \mathbb{F}_{p^2}$. We define a set-valued functor $\mathcal{N}$ on (Nilp) which sends a scheme $S \in$ (Nilp) to the set of isomorphism classes of tuples $X = (X, \iota_X, \lambda_X, \rho_X)$. Here (X, ι_X, λ_X) is a unitary p-divisible group over S of signature $(r, n-r)$ and ρ_X is an $\mathcal{O}_K$-linear quasi-isogeny

$$\rho_X \colon X \times_S \bar{S} \longrightarrow \mathbf{X} \times_{\mathrm{Spec}(\mathbb{F}_{p^2})} \bar{S}$$

such that ${}^t\rho_X \circ \lambda_X \circ \rho_X$ is a $\mathbb{Q}_p^\times$-multiple of λ_X in $\mathrm{Hom}_{\mathcal{O}_K}(X, {}^tX) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

By [RZ] Corollary 3.40 the functor $\mathcal{N}$ is representable by a formal scheme over $\mathrm{Spf}(\mathbb{Z}_{p^2})$ which is locally formally of finite type. It follows from loc. cit. Proposition 2.32 that every irreducible component of $\mathcal{N}_{\mathrm{red}}$ is projective, in particular $\mathcal{N}$ is separated.

If X is an S-valued point of $\mathcal{N}$, the height of ρ_X (considered as locally constant function on S) is a multiple of n by [Vo] 1.7 and we obtain a decomposition

$$\mathcal{N} = \coprod_{i \in \mathbb{Z}} \mathcal{N}_i,$$

where $\mathcal{N}_i$ is the open and closed formal subscheme of $\mathcal{N}$ where ρ_X has height ni. Then by [Vo] 1.8 and 1.22 we have

$$\mathcal{N}_i \neq \emptyset \iff ni \text{ is even.} \tag{1.5.1}$$

Note that in this case we have $\mathcal{N}_i \cong \mathcal{N}_0$ by Proposition 1.1 below.

(1.6) The group J of automorphisms of $\mathbf{X}$.

Let J be the algebraic group of automorphisms of the unitary isocrystal $\mathbf{N}$, i.e., for any $\mathbb{Q}_p$-algebra R we denote by $J(R) = J_{\mathbf{N}}(R)$ the group of $(K \otimes_{\mathbb{Q}_p} \mathbb{Q}_{p^2}) \otimes_{\mathbb{Q}_p} R$-linear symplectic similitudes of $\mathbf{N} \otimes R$ which commute with $\mathbf{F} \otimes \mathrm{id}_R$. This is a reductive group over $\mathbb{Q}_p$ which is an inner form of G. By Dieudonné theory, $J(\mathbb{Q}_p)$ is nothing but the group of quasi-isogenies $\mathbf{X} \to \mathbf{X}$ and therefore $J(\mathbb{Q}_p)$ acts on $\mathcal{N}$.

As we assumed that $\mathbf{N}$ is generated by elements n such that $\mathbf{F}^2 n = pn$, the arguments of [Vo] 1.19 show that if k is any perfect field extension of $\mathbb{F}_{p^2}$ the group of automorphisms of the unitary isocrystal associated with $\mathbf{X} \otimes_{\mathbb{F}_{p^2}} k$ is again J.

We fix an element $\delta \in \mathbb{Z}_{p^2}^\times$ such that $\delta^\sigma = -\delta$ and define a nondegenerate σ-hermitian form on the $\mathbb{Q}_{p^2}$-vector space $\mathbf{N}_0$ by

$$\{x, y\} := \delta \langle x, Fy \rangle. \tag{1.6.1}$$

The K-linearity of $g \in J(R)$ implies that $g(\mathbf{N}_0 \otimes R) = \mathbf{N}_0 \otimes R$ and restricting g to $\mathbf{N}_0 \otimes R$ defines an isomorphism of J with the algebraic group $\mathrm{GU}(\mathbf{N}_0, \{\,,\})$ of unitary similitudes of the hermitian space $(\mathbf{N}_0, \{\,,\})$.

We set

$$
(1.6.2) \qquad T_{\mathrm{odd}} := \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}, \qquad T_{\mathrm{even}} := \begin{pmatrix} & & & p \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{pmatrix}.
$$

By [Vo] Proposition 1.18 (see also the proof of loc. cit. Lemma 1.20) there exists a $\mathbb{Q}_{p^2}$-basis of $\mathbf{N}_0$ such that the hermitian form $\{\,,\}$ is given by T_{odd} if n is odd and by T_{even} if n is even.

Finally, we recall from [Vo] 1.22:

Proposition 1.1. *Let $i \in \mathbb{Z}$ be an integer such that ni is even (i.e. $\mathcal{N}_i \neq \emptyset$). Then there exists a $g \in J(\mathbb{Q}_p)$ such that $g(\mathcal{N}_i) = \mathcal{N}_0$. In particular, $\mathcal{N}_i$ is isomorphic to $\mathcal{N}_0$.*

(1.7) Description of the points of $\mathcal{N}$.

Let k be a perfect field extension of $\mathbb{F}_{p^2}$. Denote by $\mathbf{N}_k$ the unitary isocrystal $\mathbf{N} \otimes_{\mathbb{Z}_{p^2}} W(k)$. Then the remarks in (1.3) show that by covariant Dieudonné theory we obtain a bijection between the set $\mathcal{N}_i(k)$ and the set of $W(k)$-lattices M in $\mathbf{N}_k$ such that

- M is stable under $\mathcal{F}$ and $\mathcal{V}$,
- $M = M_0 \oplus M_1$ where $M_j = M \cap (\mathbf{N}_j)_k$,
- $\dim_k(M_0/\mathcal{V}M_1) = r$ and $\dim_k(M_1/\mathcal{V}M_0) = n - r$,
- $M = p^i M^{\perp}$, where $M^{\perp} = \{\, x \in \mathbf{N}_k \mid \langle x, M\rangle \subset W(k)\,\}$.

2 EO-stratification of $\mathcal{N}$

From now on we assume that $r = 1$.

(2.1) Examples of unitary Dieudonné spaces.

We give two examples SS and $\mathbb{B}(d)$ ($d \geq 1$ an integer) of unitary Dieudonné spaces over $\mathbb{F}_{p^2}$ which will play an important role in the Ekedahl-Oort stratification of $\mathcal{N}$.

The underlying $\mathbb{F}_{p^2}$-vector space of SS is the space of dimension 2 generated by two elements g and h. We set $M_0 = \mathbb{F}_{p^2} g$ and $M_1 = \mathbb{F}_{p^2} h$. The alternating form is given by $\langle h, g\rangle = 1$, and we have $\mathcal{F}(h) = -g$ and $\mathcal{V}(g) = h$. Then SS is a unitary Dieudonné space of signature $(0, 1)$. We indicate the definition of SS by the following diagram

$$
\begin{tikzcd}
g \\
h \arrow[u, dashed, "-"] \arrow[u, bend right]
\end{tikzcd}
$$

where $\mathcal{F}$ is given by the broken arrow and $\mathcal{V}$ by the solid arrow. The vectors generating M_0 are given in the first line and the vectors generating M_1 are given in the second line. Moreover $\langle\,,\rangle$ induces a perfect pairing on the space generated by the two vectors in the same column.

For $\mathbb{B}(d)$ the underlying graded $\mathbb{F}_{p^2}$-vector space $M = M_0 \oplus M_1$ is given by

$$M_0 = \mathbb{F}_{p^2} e_1 \oplus \ldots \oplus \mathbb{F}_{p^2} e_d, \qquad M_1 = \mathbb{F}_{p^2} f_1 \oplus \ldots \oplus \mathbb{F}_{p^2} f_d.$$

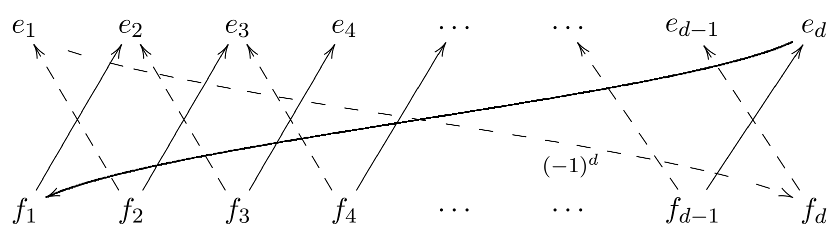
The alternating form is defined by

$$\langle e_i, f_j \rangle = (-1)^i \delta_{ij}.$$

Finally $\mathcal{F}$ and $\mathcal{V}$ are given by

$$\begin{aligned}
\mathcal{V}(f_i) &= e_{i+1}, && \text{for } i = 1, \ldots, d-1,\\
\mathcal{V}(e_d) &= f_1, &&\\
\mathcal{F}(f_i) &= e_{i-1}, && \text{for } i = 2, \ldots, d,\\
\mathcal{F}(e_1) &= (-1)^d f_d. &&
\end{aligned}$$

This is a unitary Dieudonné module of signature $(1, d-1)$. With the same convention as above $\mathbb{B}(d)$ is given by the diagram



.

With these definitions SS and $\mathbb{B}(d)$ are unitary Dieudonné spaces of signature $(0, 1)$ and $(1, d-1)$, respectively.

(2.2) Classification of unitary Dieudonné spaces.

We call a unitary Dieudonné module M over a perfect field k *supersingular*, if its isocrystal $(M[\frac{1}{p}], \mathcal{F})$ is supersingular.

By [BW] (4.2) and (3.6) we have the following classification result.

Theorem 2.1. *Let k be algebraically closed and let $\bar{M}$ be a unitary Dieudonné space of signature $(1, n-1)$. The following assertions are equivalent.*

(1) *There exists a supersingular unitary Dieudonné module M such that $\bar{M} \cong M \otimes_{W(k)} k$.*

(2) *Every unitary Dieudonné module M such that $\bar{M} \cong M \otimes_{W(k)} k$ is supersingular.*

(3) *There exists a (necessarily unique) integer* $0 \leq \sigma \leq (n-1)/2$ *such that*

$$\bar{M} \cong (\mathbb{B}(2\sigma+1) \oplus SS^{n-(2\sigma+1)}) \otimes_{\mathbb{F}_{p^2}} k,$$

where $\mathbb{B}(2\sigma{+}1)$ *and* SS *are the unitary Dieudonné spaces defined in* (2.1).

We call $\bar{M}$ *supersingular* if it satisfies the equivalent conditions of Theorem 2.1. We set

$$\mathbb{M}_\sigma := \mathbb{B}(2\sigma+1) \oplus SS^{n-(2\sigma+1)} \tag{2.2.1}$$

and we denote by $\overline{\mathbb{X}}_\sigma$ the corresponding at level 1 truncated unitary Barsotti-Tate group.

(2.3) Ekedahl-Oort strata of $\mathcal{N} \otimes_{\mathbb{Z}_{p^2}} \mathbb{F}_{p^2}$.

Fix an integer $0 \leq \sigma \leq (n-1)/2$. The Ekedahl-Oort stratum $\mathcal{N}(\sigma)$ in $\mathcal{N} \otimes_{\mathbb{Z}_{p^2}} \mathbb{F}_{p^2}$ is defined as follows. For any $\mathbb{F}_{p^2}$-scheme S we define $\mathcal{N}(\sigma)(S)$ to be the $X \in \mathcal{N}(S)$ such that $X[p]$ is locally for the fppf-topology isomorphic to $(\overline{\mathbb{X}}_\sigma)_S$.

By [Wd] (6.7) the monomorphism $\mathcal{N}(\sigma) \hookrightarrow \mathcal{N} \otimes_{\mathbb{Z}_{p^2}} \mathbb{F}_{p^2}$ is representable by an immersion. In particular $\mathcal{N}(\sigma)$ is a formal scheme over $\mathbb{F}_{p^2}$, locally formally of finite type. These formal schemes are called the *Ekedahl-Oort strata* of $\mathcal{N} \otimes_{\mathbb{Z}_{p^2}} \mathbb{F}_{p^2}$. For any perfect extension k of $\mathbb{F}_{p^2}$ we have

$$\mathcal{N}(k) = \biguplus_{0 \leq \sigma \leq \frac{n-1}{2}} \mathcal{N}(\sigma)(k).$$

by Theorem 2.1.

Theorem 2.2. *The Ekedahl-Oort strata* $\mathcal{N}(\sigma)$ *are formally smooth over* $\mathbb{F}_{p^2}$.

Proof. Let R be an $\mathbb{F}_{p^2}$-algebra. Associating to R-valued points (X, ρ_X) of $\mathcal{N}$ its unitary BT_1 $X[p]$ defines a morphism

$$\alpha\colon \mathcal{N} \otimes \mathbb{F}_{p^2} \to \mathscr{B},$$

where $\mathscr{B}$ is the algebraic stack that classifies unitary BT_1 of signature $(1, n-1)$ (cf. [Wd] (1.8)). In $\mathscr{B}$ there is the smooth locally closed substack $\mathscr{B}(\sigma)$ classifying unitary BT_1 which are locally for the fppf-topology isomorphic to $\overline{\mathbb{X}}_\sigma[p]$. By definition $\mathcal{N}(\sigma)$ is the inverse image of $\mathscr{B}(\sigma)$ under α. Therefore it suffices to show that α is smooth.

By [Wd] (2.17), associating to a unitary p-divisible group X its p-torsion $X[p]$ is formally smooth. Therefore the smoothness of α follows from Drinfeld's theorem that quasi-isogenies can be always deformed uniquely. $\square$

(2.4) Superspecial gap.

We will now connect the Ekedahl-Oort stratification with the notion of type defined in [Vo].

We fix an integer $i \in \mathbb{Z}$ such that ni is even (1.5.1). Let k be a perfect field extension of $\mathbb{F}_{p^2}$ and let $x \in \mathcal{N}_i(k)$. Let M be the unitary Dieudonné module over k of signature $(1, n-1)$ corresponding to x (1.7) and let $N = M \otimes_{\mathbb{Z}} \mathbb{Q}$ be its isocrystal. We set $\tau := p^{-1}\mathcal{F}^2 \colon N \to N$ and define

$$\Lambda^+(M) := \sum_{l \geq 0} \tau^l(M), \qquad \Lambda^-(M) := \bigcap_{l \geq 0} \tau^l(M).$$

These are $\mathcal{O}_K \otimes W(k)$-submodules and therefore there is a decomposition $\Lambda^+(M) = \Lambda^+(M)_0 \oplus \Lambda^+(M)_1$. Note that $\Lambda^+(M)_j$ is τ-invariant for $j = 0, 1$. Similar statements hold for $\Lambda^-(M)$. We set

$$\begin{aligned} \sigma^+(M_j) &:= \inf\{\, d \geq 0 \mid \Lambda^+(M)_j = \sum_{l=0}^{d} \tau^l(M_j)\,\}, \\ \sigma^-(M_j) &:= \inf\{\, d \geq 0 \mid \Lambda^-(M)_j = \bigcap_{l=0}^{d} \tau^l(M_j)\,\}. \end{aligned} \tag{2.4.1}$$

The pairing $p^{-i}\langle\,,\rangle$ induces a perfect pairing on M and therefore a perfect duality between $\Lambda^-(M)_0$ and $\Lambda^+(M)_1$ and between $\Lambda^-(M)_1$ and $\Lambda^+(M)_0$. Hence

$$\begin{gathered} \sigma^+(M_0) = \sigma^-(M_1) \leq \frac{n-1}{2}, \qquad \sigma^+(M_1) = \sigma^-(M_0) \leq \frac{n+1}{2}, \\ \sigma^+(M_j) = [\Lambda^+(M_j) : M_j], \qquad \sigma^-(M_j) = [M_j : \Lambda^-(M_j)], \\ p\Lambda^+(M) \subset \Lambda^-(M) \subset M \subset \Lambda^+(M) \subset p^{-1}\Lambda^-(M). \end{gathered} \tag{2.4.2}$$

by [Vo] Lemma 2.2. Moreover we have

$$\sigma^+(M_0) = \begin{cases} \sigma^-(M_0) = 0, & \text{if } \tau(M) = M; \\ \sigma^-(M_0) - 1, & \text{otherwise.} \end{cases}$$

Definition 2.3. *The integer $\sigma(M) := \sigma^+(M_0)$ is called the* superspecial gap *of M.*

Then $0 \leq \sigma(M) \leq \frac{n-1}{2}$ and if $\sigma(M) > 0$, we have

$$[\Lambda^+(M_0) : \Lambda^-(M_0)] = [\Lambda^+(M_1) : \Lambda^-(M_1)] = 2\sigma(M) + 1. \tag{2.4.3}$$

(2.5) Ekedahl-Oort strata and the superspecial gap.

The following theorem shows that the (scheme theoretical) Ekedahl-Oort stratification and the (set theoretical) decomposition by the superspecial gap coincide.

Theorem 2.4. *Let $x \in \mathcal{N}(k)$, where k is an algebraically closed extension of $\mathbb{F}_{p^2}$, and let M be the associated unitary Dieudonné module. Let σ be an integer with $0 \leq \sigma \leq \frac{n-1}{2}$. Then $x \in \mathcal{N}(\sigma)(k)$ if and only if $\sigma(M) = \sigma$.*

Proof. We have to show that $\sigma(M) = \sigma$ is equivalent to $\bar{M} \cong (\mathbb{M}_\sigma)_k$ (with the notations of (2.2.1)).

For any k-vector space $\bar{U}$ of $\bar{M}_1$ we set

$$\bar{\tau}(\bar{U}) := \mathcal{V}^{-1}(\mathcal{F}(\bar{U})) \cap \bar{M}_1,$$

where $\mathcal{V}^{-1}(\ )$ denotes the preimage under $\mathcal{V}$. We also lift $\bar{\tau}$ by setting

$$\tau'(U) := (\tau(U) \cap M_1) + pM_1$$

for any $W(k)$-submodule U of M_1 with $pM_1 \subset U$. By (2.4.2) we have $\tau^d(M_1) \supset pM_1$ for all $d \geq 0$ and hence

$$(\tau')^d(M_1) = \bigcap_{i=0}^{d} \tau^i(M_1).$$

An easy calculation using the explicit definition of $\mathbb{B}(r)$ and SS shows that $\bar{M} \cong (\mathbb{M}_\sigma)_k$ if and only if

$$\dim(\bar{\tau}^i(\bar{M}_1)) = \begin{cases} \dim_k(\bar{M}_1) - i, & i \leq \sigma; \\ \dim_k(\bar{M}_1) - \sigma, & i \geq \sigma. \end{cases}$$

Hence $\bar{M}$ and $(\mathbb{M}_\sigma)_k$ are isomorphic if and only if σ is the smallest integer $d \geq 0$ such that

$$\bigcap_{i=0}^{d} \tau^i(M_1) = \bigcap_{i=0}^{d+1} \tau^i(M_1).$$

But by (2.4.2) we have $\sigma(M) = \sigma^-(M_1)$ and therefore $\sigma(M)$ is the smallest integer $d \geq 0$ such that $M_1 \cap \tau(M_1) \cap \cdots \cap \tau^d(M_1)$ is τ-invariant. This proves the theorem. $\square$

3 Subschemes of $\mathcal{N}$ indexed by vertices of the Bruhat-Tits building of J

In this section we attach to each vertex Λ of the Bruhat-Tits building of J a subschemes $\mathcal{N}_\Lambda$ of $\mathcal{N}$ and prove that $\mathcal{N}_\Lambda$ is isomorphic to a (generalized) Deligne-Lusztig variety.

(3.1) Vertices of the building of J.

We recall now some definitions and results of [Vo]. If $\Lambda \subset \mathbf{N}_0$ is a $\mathbb{Z}_{p^2}$-lattice we define

$$\Lambda^\vee = \{\, x \in \mathbf{N}_0 \mid \{x, \Lambda\} \subset \mathbb{Z}_{p^2} \,\},$$

where $\{\, ,\}$ is the hermitian form on $\mathbf{N}_0$ defined in (1.6.1). Fix $i \in \mathbb{Z}$ such that ni is even and set

$$\mathcal{L}_i := \{\, \Lambda \subset \mathbf{N}_0 \ \mathbb{Z}_{p^2}\text{-lattice} \mid p^{i+1}\Lambda^\vee \subsetneq \Lambda \subset p^i\Lambda^\vee \,\}.$$

We construct from $\mathcal{L}_i$ an abstract simplicial complex $\mathcal{B}_i$. An m-simplex of $\mathcal{B}_i$ is a subset $S \subset \mathcal{L}_i$ of $m+1$ elements which satisfies the following condition. There exists an ordering $\Lambda_0, \dots, \Lambda_m$ of the elements of S such that

$$p^{i+1}\Lambda_m^\vee \subsetneq \Lambda_0 \subsetneq \Lambda_1 \subsetneq \cdots \subsetneq \Lambda_m.$$

Let $\tilde{J} = \mathrm{SU}(\mathbf{N}_0, \{\, ,\})$ be the derived group of J. Note that $\tilde{J}$ is a simply connected semisimple algebraic group over $\mathbb{Q}_p$. There is an obvious action of $\tilde{J}(\mathbb{Q}_p)$ on $\mathcal{L}_i$. We denote by $\mathcal{B}(\tilde{J}, \mathbb{Q}_p)$ the abstract simplicial complex of the Bruhat-Tits building of $\tilde{J}$. By [Vo] Theorem 3.6 we have a natural identification of $\mathcal{B}_i$ (for fixed i) with $\mathcal{B}(\tilde{J}, \mathbb{Q}_p)$. In particular we can identify $\mathcal{L}_i$ with the set of vertices of $\mathcal{B}(\tilde{J}, \mathbb{Q}_p)$. This identification is $\tilde{J}(\mathbb{Q}_p)$-equivariant.

For $\Lambda \in \mathcal{L}_i$ the index of $p^{i+1}\Lambda^\vee$ in Λ is always an odd number, say $2d+1$ by [Vo] Remark 2.4. We call the integer d the *orbit type of* Λ and denote it by $t(\Lambda)$. We always have $0 \leq d \leq \frac{n-1}{2}$ and for every such d there exists a $\Lambda \in \mathcal{L}_i$ such that $t(\Lambda) = d$ (loc. cit.).

Our terminology is explained by the following remark (which follows from [Vo] 1.17).

Remark 3.1. *Two lattices of $\mathcal{L}_i$ have the same orbit type if and only if they are in the same $\tilde{J}(\mathbb{Q}_p)$-orbit.*

(3.2) p-divisible groups attached to a vertex.

Fix again $i \in \mathbb{Z}$ such that ni is even. For each $\Lambda \in \mathcal{L}_i$ we will construct two p-divisible O_K-modules X_{Λ^+} and X_{Λ^-} over $\mathbb{F}_{p^2}$ with O_K-linear polarization λ_{Λ^+} and λ_{Λ^-}, respectively. Both will be equipped with an O_K-linear quasi-isogeny

$$\rho_{\Lambda^\pm} \colon X_{\Lambda^\pm} \to \mathbf{X}$$

which is compatible with the polarizations up to a power of p.

For this we define lattices

$$\begin{aligned}
\Lambda_0^+ &:= \Lambda, \\
\Lambda_1^+ &:= \mathbf{V}^{-1}(\Lambda_0^+) \\
\Lambda^+ &:= \Lambda_0^+ \oplus \Lambda_1^+, \\
\Lambda^- &:= \{\, x \in \mathbf{N} \mid p^{-i}\langle x, \Lambda^+\rangle \subset \mathbb{Z}_{p^2} \,\} = p^i(\Lambda^+)^\perp.
\end{aligned} \tag{3.2.1}$$

As $\mathbf{F} = \mathbf{V}$ it follows immediately from the definitions that $\Lambda^{\pm}$ are Dieudonné submodules of the isocrystal $\mathbf{N}$. Clearly $\Lambda^{\pm}_j = \Lambda^{\pm} \cap \mathbf{N}_j$ and therefore the K-action on $\mathbf{N}$ restricts to an O_K-action on $\Lambda^{\pm}$. As $\Lambda \subset p^i\Lambda^{\vee}$, the pairing $p^{-i+1}\langle\,,\rangle$ on $\mathbf{N}$ induces a (not necessarily perfect) $\mathbb{Z}_{p^2}$-pairing on $\Lambda^{\pm}$. Hence the unitary p-divisible groups $X_{\Lambda^{\pm}}$ associated to the unitary Dieudonné modules $\Lambda^{\pm}$ have all the desired properties. The definition of Λ^+ shows that X_{Λ^+} is a p-divisible O_K-module of signature $(0,n)$ and an easy calculation shows that this holds for X_{Λ^-}, too.

Note that we have

$$(3.2.2)\quad \Lambda^-_0 = \{\, x \in \mathbf{N}_0 \mid p^i\langle x, \mathbf{V}^{-1}\Lambda\rangle = p^{-i-1}\{x,\Lambda\} \subset \mathbb{Z}_{p^2}\,\} = p^{i+1}(\Lambda^+_0)^{\vee}$$

and hence

$$(3.2.3)\qquad [\Lambda^+_0 : \Lambda^-_0] = [\Lambda^+_1 : \Lambda^-_1] = 2t(\Lambda) + 1.$$

From this it follows that if $\tilde{\Lambda} \in \mathcal{L}_i$ is a second lattice with $\tilde{\Lambda} \subsetneq \Lambda$ we have

$$(3.2.4)\qquad t(\tilde{\Lambda}) < t(\Lambda).$$

By definition, $p^{-i}\langle\,,\rangle$ induces a perfect duality between Λ^+ and Λ^-. This duality defines an isomorphism of p-divisible O_K-modules $X_{\Lambda^+} \xrightarrow{\sim} {}^tX_{\Lambda^-}$ which makes the diagram

$$(3.2.5)\qquad \begin{array}{ccc} X_{\Lambda^+} & \xrightarrow{\sim} & {}^tX_{\Lambda^-} \\ {\scriptstyle\rho_{\Lambda^+}}\downarrow & & \uparrow{\scriptstyle {}^t\rho_{\Lambda^-}} \\ \mathbf{X} & \xrightarrow[\lambda_{\mathbf{X}}]{\sim} & {}^t\mathbf{X} \end{array}$$

commutative. We set

$$(3.2.6)\qquad \mathbb{B}_{\Lambda} := \Lambda^+/\Lambda^-.$$

As $p\Lambda^+ \subset \Lambda^-$ and $p^i(\Lambda^+)^{\perp} = \Lambda^-$, $\mathbb{B}_{\Lambda}$ carries the structure of a unitary Dieudonné space over $\mathbb{F}_{p^2}$, where the alternating form is induced by $p^{-i+1}\langle\,,\rangle$.

(3.3) Schemes $\mathcal{N}_{\Lambda}$ attached to a vertex Λ.

We fix $\Lambda \in \mathcal{L}_i$. For any $\mathbb{F}_{p^2}$-scheme and for $(X, \rho_X) \in \mathcal{N}_i(S)$ we define quasi-isogenies

$$\rho_{X,\Lambda^+}\colon X \xrightarrow{\rho_X} \mathbf{X}_S \xrightarrow{(\rho_{\Lambda^+})_S^{-1}} (X_{\Lambda^+})_S,$$
$$\rho_{\Lambda^-,X}\colon (X_{\Lambda^-})_S \xrightarrow{(\rho_{\Lambda^-})_S} \mathbf{X} \xrightarrow{\rho_X^{-1}} X$$

Let $t(\Lambda)$ be the orbit type of Λ. It follows from (3.2.5) and (3.2.3) that

$$\text{height}(\rho_{X,\Lambda^+}) = \text{height}(\rho_{\Lambda^-,X}) = 2t(\Lambda) + 1 \tag{3.3.1}$$

and that ρ_{X,Λ^+} is an isogeny if and only if $\rho_{\Lambda^-,X}$ is an isogeny.

Let $\mathcal{N}_\Lambda$ be the subfunctor of $\mathcal{N}_i \otimes_{\mathbb{Z}_{p^2}} \mathbb{F}_{p^2}$ consisting of those points $(X, \rho_X) \in \mathcal{N}_i(S)$ such that ρ_{X,Λ^+} or, equivalently, $\rho_{\Lambda^-,X}$ is an isogeny.

Lemma 3.2. *The functor $\mathcal{N}_\Lambda$ is representable by a projective $\mathbb{F}_{p^2}$-scheme and the monomorphism $\mathcal{N}_\Lambda \hookrightarrow \mathcal{N}_i$ is a closed immersion.*

Proof. We set $d := t(\Lambda)$. First of all $\mathcal{N}_\Lambda \hookrightarrow \mathcal{N}_i$ is a closed immersion by [RZ] 2.9. Let R be any $\mathbb{F}_{p^2}$-algebra. Associating to $(X, \rho_X) \in \mathcal{N}_\Lambda(R)$ the kernel of $\rho_{\Lambda^-,X}$ defines a morphism of $\mathcal{N}_\Lambda$ to the functor $\mathscr{F}$ on $\mathbb{F}_{p^2}$-algebras R such that $\mathscr{F}(R)$ is the set of O_K-subgroup schemes $\mathcal{G}$ of $X_{\Lambda^-} \otimes R$ of height $2d+1$ such that the polarization λ_{Λ^-} on X_{Λ^-} induces an isomorphism on $(X_{\Lambda^-} \otimes R)/\mathcal{G}$. Conversely, sending $\mathcal{G} \in \mathscr{F}(R)$ to the pair consisting of $X := (X_{\Lambda^-} \otimes R)/\mathcal{G}$ and the inverse ρ_X of quasi-isogeny

$$\mathbf{X} \otimes R \xrightarrow{\rho^{-1}_{\Lambda^-,R}} X_{\Lambda^-} \otimes R \longrightarrow X$$

defines an inverse morphism of functors. Therefore $\mathcal{N}_\Lambda$ and $\mathscr{F}$ are isomorphic.

The functor $\mathscr{F}$ is a closed subfunctor of the functor parametrizing subgroup schemes of height $2d+1$ of X_{Λ^-} which is clearly representable by a projective scheme over $\mathbb{F}_{p^2}$. Therefore $\mathscr{F}$ and hence $\mathcal{N}_\Lambda$ are representable by projective $\mathbb{F}_{p^2}$-schemes. □

Lemma 3.3. *Let k be an algebraically closed extension of $\mathbb{F}_{p^2}$. For any lattice Γ in $\mathbf{N}$ we set $\Gamma_k = \Gamma \otimes_{\mathbb{Z}_{p^2}} W(k)$. For $x \in \mathcal{N}_i(k)$ denote by $M \subset \mathbf{N}_k$ the corresponding unitary Dieudonné module. Then the following assertions are equivalent.*
(1) *$x \in \mathcal{N}_\Lambda(k)$.*
(2) *$M \subset (\Lambda^+)_k$.*
(3) *$\Lambda^+(M) \subset (\Lambda^+)_k$.*
(4) *$M_0 \subset \Lambda_k$.*

Proof. The equivalence of (1) and (2) is clear from the definition of $\mathcal{N}_\Lambda$, and the inclusion $M \subset \Lambda^+(M)$ shows that (3) implies (2). As $(\Lambda^+)_k$ is τ-invariant and $\Lambda^+(M)$ is the smallest τ-invariant lattice of $\mathbf{N}_k$ containing M we see that (2) also implies (3). Clearly (2) implies (4) (recall that $\Lambda = \Lambda_0^+$ by definition). Conversely, if $M_0 \subset \Lambda_k$, we have $M_1 \subset p^{-1}\mathcal{F}(M_0) \subset p^{-1}\mathbf{F}(\Lambda)_k = (\Lambda_1^+)_k$. □

This lemma shows that for any algebraically closed extension k of $\mathbb{F}_{p^2}$ we have

$$\mathcal{N}_\Lambda(k) = \mathcal{V}(\Lambda)(k), \tag{3.3.2}$$

where $\mathcal{V}(\Lambda)(k)$ is the subset of $\mathcal{N}_i(k)$ defined in [Vo] 2.3.

We will now define a Deligne-Lusztig variety Y_Λ for a certain reductive group $\tilde{J}_\Lambda$ over $\mathbb{F}_p$. Below we will show that Y_Λ and $\mathcal{N}_\Lambda$ are isomorphic. For this we first recall some general facts about Deligne-Lusztig varieties.

(3.4) Deligne-Lusztig varieties.

Let k be a finite field, let G be a connected reductive group over k and let (W, S) be the Weyl system of $G_{\bar{k}}$, where $\bar{k}$ is a fixed algebraic closure of k. Let $F\colon G \to G$ be the Frobenius morphism with respect to k. Then F acts by automorphisms on W. As G is quasi-split, $F(S) = S$. For any two subsets $I, I' \subset S$ we denote by ${}^I W^{I'}$ the set of $w \in W$ such that w is the (necessarily unique) element of minimal length in its double coset $W_I w W_{I'}$. Here W_I is the subgroup of W generated by I (and similar for $W_{I'}$).

For any subset $I \subset S$ we denote by $\mathrm{Par}_{G,I} = \mathrm{Par}_I$ the scheme of parabolic subgroups of G of type I. It is defined over the extension of degree r of k in $\bar{k}$, where r is the minimal integer $r \geq 1$ such that $F^r(I) = I$. We denote this field by $\kappa(I)$. Clearly, we have $\kappa(I) = \kappa(F(I))$.

Let κ be the compositum of $\kappa(I)$ and $\kappa(I')$ in $\bar{k}$. The group G_κ acts diagonally on $\mathrm{Par}_{I,\kappa} \times \mathrm{Par}_{I',\kappa}$. The orbits are in natural bijection to ${}^I W^{I'}$ and we denote by $\mathcal{O}_{I,I'}(w)$ the orbit corresponding to $w \in {}^I W^{I'}$. Clearly, $\mathcal{O}_{I,I'}(w)$ is geometrically irreducible and smooth. Moreover, it is easy to check that

$$\dim(\mathcal{O}_{I,I'}(w)) = \ell(w) + \dim \mathrm{Par}_{I \cap I'}\,.$$

Fix $I \subset S$ and $w \in {}^I W^{F(I)}$. We define the *Deligne-Lusztig variety* $X_I(w)$ as the (locally closed) subscheme of Par_I parametrizing parabolic subgroups $P \in \mathrm{Par}_I$ such that P and $F(P) \in \mathrm{Par}_{F(I)}$ are in relative position w.

By definition, $X_I(w)$ is the intersection of $\mathcal{O}_{I,F(I)}(w)$ and the graph of the Frobenius

$$\Gamma_F\colon\ \mathrm{Par}_I \to \mathrm{Par}_I \times \mathrm{Par}_{F(I)}, \qquad P \mapsto (P, F(P)).$$

It is easily checked that this intersection is transversal. Therefore $X_I(w)$ is smooth of pure dimension

$$\dim(X_I(w)) = \ell(w) + \dim \mathrm{Par}_{I \cap F(I)} - \dim(\mathrm{Par}_I). \tag{3.4.1}$$

Note that $X_I(\mathrm{id})$ is projective.

Finally, we recall the following theorem of Bonnafé and Rouquier [BR].

Theorem 3.4. *The following assertions are equivalent.*
(i) *$X_I(w)$ is geometrically irreducible.*
(ii) *$X_I(w)$ is connected.*
(iii) *There exists no $J \subsetneq S$ with $F(J) = J$ and with $W_I w \subset W_J$.*

Proof. The implication "(iii) $\Rightarrow$ (i)" is proved in loc. cit. and "(i) $\Rightarrow$ (ii)" is trivial. Moreover, the proof of Bonnafé and Rouquier (their "first step") shows in facts the implication "(ii) $\Rightarrow$ (iii)". $\square$

Thus $X_I(\mathrm{id})$ is geometrically irreducible if and only if $\bigcup_{r\geq 1} F^r(I) = S$.

(3.5) The Deligne-Lusztig variety Y_Λ.

Let $\Lambda \in \mathcal{L}_i$ be a lattice of orbit type $d := t(\Lambda)$. We set $V_\Lambda = \Lambda_0^+/\Lambda_0^-$ and we endow V_Λ with the hermitian pairing $(\,,) = (\,,)_\Lambda$ induced by $p^{-i}\{\,,\}$. This is a well defined and nondegenerate pairing by (3.2.2). Therefore $(V_\Lambda, (\,,))$ is a hermitian space V_Λ over $\mathbb{F}_{p^2}$ of dimension $2d+1$.

Set $\tilde{J}_\Lambda = \mathrm{SU}(V_\Lambda, (\,,))$ which is a connected reductive group over $\mathbb{F}_p$. We remark that $\tilde{J}_\Lambda$ is the maximal reductive quotient of the special fibre of the smooth affine group scheme attached to the vertex of $\mathcal{B}(\tilde{J}, \mathbb{Q}_p)$ which corresponds to Λ. We will not use this remark in the sequel.

We denote by $F\colon \tilde{J}_\Lambda \to \tilde{J}_\Lambda$ the Frobenius morphism over $\mathbb{F}_p$ and by (W, S) be the Weyl system of $\tilde{J}_\Lambda$. Then F induces an automorphism of W which we again denote by F. As

$$\tilde{J}_\Lambda \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2} \cong \mathrm{SL}(V_\Lambda) \cong \mathrm{SL}_{2d+1,\mathbb{F}_{p^2}},$$

we can identify (W, S) with $(S_{2d+1}, \{s_1, \dots, s_{2d}\})$, where S_{2d+1} is the symmetric group of $\{1, \dots, 2d+1\}$ and where s_i is the transposition of i and $i+1$. The automorphism F of (W, S) is induced by the unique nontrivial automorphism of the Dynkin diagram of $\tilde{J}_\Lambda$, i.e. F is given by the conjugation with w_0, where w_0 is the element of maximal length in S_{2d+1} (i.e. $w_0(i) = 2d+2-i$ for all $i = 1, \dots, 2d+1$).

For any subset $I \subset S$, the scheme $\mathrm{Par}_I = \mathrm{Par}_{\tilde{J}_\Lambda, I}$ of parabolic subgroups of $\tilde{J}_\Lambda$ of type I is defined over $\mathbb{F}_p$ if and only if $I = F(I)$, otherwise it is defined over $\mathbb{F}_{p^2}$. We now set

$$I_\Lambda := \{s_1, \dots, s_d, s_{d+2}, \dots, s_{2d}\}.$$

Then $\kappa(I_\Lambda) = \mathbb{F}_{p^2}$. Via the isomorphism $\tilde{J}_\Lambda \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2} \cong \mathrm{SL}(V_\Lambda)$ we see that Par_{I_Λ} is the $\mathbb{F}_{p^2}$-scheme which parametrizes $(d+1)$-dimensional subspaces of the $\mathbb{F}_{p^2}$-vector space V_Λ.

We set

$$Y_\Lambda := X_{I_\Lambda}(\mathrm{id}).$$

By [Vo] Lemma 2.17 this definition coincides with the definition given in loc. cit. 2.12.

Lemma 3.5. *The $\mathbb{F}_{p^2}$-scheme Y_Λ is projective, smooth, and geometrically irreducible of dimension $d = t(\Lambda)$.*

Proof. This is proved in [Vo] Proposition 2.18 and Theorem 2.21, but it follows also from general facts on Deligne-Lusztig varieties recalled in (3.4). □

(3.6) Modules associated with some isogenies.

To construct an isomorphism $\mathcal{N}_\Lambda \xrightarrow{\sim} Y_\Lambda$ we will use the Lie algebra of the universal vector extension (see [Me]). Let S be an $\mathbb{F}_p$-scheme. Recall that sending a p-divisible group to the Lie algebra of its universal vector extension defines a functor $X \mapsto D(X)$ from the category of p-divisible groups over S to the category of locally free $\mathscr{O}_S$-modules. This functor is compatible with base change $S' \to S$ and we have $\mathrm{height}(X) = \mathrm{rk}_{\mathscr{O}_S}(D(X))$. Moreover the Hodge filtration is a locally direct summand $H(X)$ of $D(X)$ whose corank is equal to $\mathrm{rk}_{\mathscr{O}_S}(\mathrm{Lie}(X))$.

If S is the spectrum of a perfect field k, there is a functorial isomorphism $D(X) \cong M(X)/pM(X) =: \bar{M}(X)$, where $M(X)$ is the underlying $W(k)$-module of the covariant Dieudonné module of X and we have $H(X) = \mathcal{V}(\bar{M}(X))$.

Proposition 3.6. *Let $\rho\colon X \to Y$ be a homomorphism of p-divisible groups over S with $\mathrm{Ker}(\rho) \subset X[p]$. Then $\mathrm{Coker}(D(\rho))$ is a locally free $\mathscr{O}_S$-module and $\mathrm{rk}_{\mathscr{O}_S}(\mathrm{Coker}(D(\rho)) = \mathrm{height}(\rho)$.*

Before giving the proof we recall two lemmas.

Lemma 3.7. *Let S be any scheme and let $\mathscr{F}^1 \to \mathscr{F}^2 \to \mathscr{F}^3$ be a complex of finitely presented $\mathscr{O}_S$-modules such that $\mathscr{F}^2$ and $\mathscr{F}^3$ are locally free. Then $\mathscr{F}^\bullet$ is exact if and only if for each geometric point $\bar{s} \to S$ the complex $\mathscr{F}^\bullet \otimes_{\mathscr{O}_S} \kappa(\bar{s})$ is exact.*

Proof. This is a special case of [EGA] IV, 12.3.3. □

Lemma 3.8. *Let R be a local Artinian ring with residue field k and let M be a finitely generated R-module. Then*

$$\mathrm{lg}_R(M) \leq \mathrm{lg}_R(R)\dim_k(M \otimes_R k).$$

Moreover, equality holds if and only if M is a free module.

Proof. This is an immediate application of the Lemma of Nakayama. $\square$

Proof. (of the Proposition) If S is the spectrum of a perfect field k, we have $\dim_k(\mathrm{Coker}(D(\rho))) = \mathrm{height}(\rho)$ by covariant Dieudonné theory. Now for an abritrary $\mathbb{F}_p$-scheme S the formation of $\mathrm{Coker}(D(\rho))$ is compatible with base change $S' \to S$ and therefore it suffices to show that $\mathrm{Coker}(D(\rho))$ is locally free.

As $\mathrm{Ker}(\rho) \subset X[p]$ there exists a unique isogeny $\rho'\colon Y \to X$ such that $\rho' \circ \rho = p\,\mathrm{id}_X$ and $\rho \circ \rho' = p\,\mathrm{id}_Y$. Setting $\varphi := D(\rho)$ and $\varphi' = D(\rho')$ we obtain a complex

$$\dots \xrightarrow{\varphi'} D(X) \xrightarrow{\varphi} D(Y) \xrightarrow{\varphi'} D(X) \xrightarrow{\varphi} \dots \tag{3.6.1}$$

whose formation is compatible with base change $S' \to S$. By covariant Dieudonné theory this complex is exact if S is the spectrum of a perfect field. Therefore (3.6.1) is exact for arbitrary S by Lemma 3.7.

To prove that $\mathrm{Coker}(\varphi)$ is locally free we can assume by standard arguments that S is the spectrum of a local Artin ring R. Let k be its residue field. We set

$$\begin{gathered} d := \dim_k(\mathrm{Coker}(\varphi \otimes_R \mathrm{id}_k)), \qquad d' := \dim_k(\mathrm{Coker}(\varphi' \otimes_R \mathrm{id}_k)), \\ \ell := \mathrm{lg}_R(R), \qquad h := \mathrm{rk}_R(D(X)) = \mathrm{rk}_R(D(Y)). \end{gathered}$$

Then $h = d + d'$ and $h\ell = \mathrm{lg}_R(D(X)) = \mathrm{lg}_R(D(Y))$. We have

$$\mathrm{lg}_R(\mathrm{Coker}(\varphi) = \mathrm{lg}_R(\mathrm{Ker}(\varphi)) = \mathrm{lg}_R(\mathrm{Im}(\varphi')) = h\ell - \mathrm{lg}_R(\mathrm{Coker}(\varphi')).$$

By Lemma 3.8 we have

$$h\ell = \mathrm{lg}_R(\mathrm{Coker}(\varphi) + \mathrm{lg}_R(\mathrm{Coker}(\varphi')) \le d\ell + d'\ell = h\ell$$

and therefore we must have $\mathrm{lg}_R(\mathrm{Coker}(\varphi) = d\ell$ which implies that $\mathrm{Coker}(\varphi)$ is a free R-module again by Lemma 3.8. $\square$

Corollary 3.9. *Let S be an $\mathbb{F}_p$-scheme and let $\rho_i\colon X \to Y_i$ for $i = 1, 2$ be two isogenies of p-divisible groups over S such that $\mathrm{Ker}(\rho_1) \subset \mathrm{Ker}(\rho_2) \subset X[p]$. Then $\mathrm{Ker}(D(\rho_1))$ is locally a direct summand of the locally free $\mathscr{O}_S$-module $\mathrm{Ker}(D(\rho_2))$ and the formation of $\mathrm{Ker}(D(\rho_i))$ commutes with base change $S' \to S$.*

Proof. As $\mathrm{Coker}(D(\rho_i))$ is locally free, $\mathrm{Im}(D(\rho_i))$ is locally a direct summand of $D(Y_i)$ and therefore locally free. Hence the exact sequence

$$0 \to \mathrm{Ker}(D(\rho_i)) \to D(X) \to \mathrm{Im}(D(\rho_i)) \to 0$$

locally splits and $\mathrm{Ker}(D(\rho_i))$ is locally a direct summand of $D(X)$ whose formation commutes with base change. Now the exact sequence

$$0 \to \mathrm{Ker}(D(\rho_2))/\,\mathrm{Ker}(D(\rho_1)) \longrightarrow D(X)/\,\mathrm{Ker}(D(\rho_1)) \\ \longrightarrow D(X)/\,\mathrm{Ker}(D(\rho_2)) \to 0$$

shows that $\mathrm{Ker}(D(\rho_2))/\,\mathrm{Ker}(D(\rho_1))$ is a locally free $\mathscr{O}_S$-module and therefore $\mathrm{Ker}(D(\rho_1))$ is locally a direct summand of $\mathrm{Ker}(D(\rho_2))$. □

(3.7) The isomorphism $f\colon \mathcal{N}_\Lambda \xrightarrow{\sim} Y_\Lambda$.

We will now define a morphism $f\colon \mathcal{N}_\Lambda \to Y_\Lambda$. Let R be an $\mathbb{F}_{p^2}$-algebra and let $(X, \rho_X) \in \mathcal{N}_\Lambda(R)$. By definition we have isogenies

$$X_{\Lambda^-,R} \xrightarrow{\rho_{\Lambda^-,X}} X \xrightarrow{\rho_{X,\Lambda^+}} X_{\Lambda^+,R}.$$

The composition is $\rho_\Lambda \otimes \mathrm{id}_R$, where $\rho_\Lambda\colon X_{\Lambda^-} \to X_{\Lambda^+}$ is the isogeny over $\mathbb{F}_{p^2}$ which is given via Dieudonné theory by the inclusion $\Lambda^- \subset \Lambda^+$. As $p\Lambda^+ \subset \Lambda^-$, we have $\mathrm{Ker}(\rho_\Lambda) \subset X_{\Lambda^-}[p]$ and by (3.2.6) the unitary Dieudonné space associated with $\mathrm{Ker}(\rho_\Lambda)$ is $\mathbb{B}_\Lambda$.

By Corollary 3.9, $E(X) := \mathrm{Ker}(D(\rho_{\Lambda^-,X}))$ is a direct summand of the R-module $\mathbb{B}_\Lambda \otimes_{\mathbb{F}_{p^2}} R$. Its rank is $2d+1$ by (3.3.1). The O_K-action on D and on $\mathbb{B}_\Lambda$ defines a decomposition $E(X) = E_0(X) \oplus E_1(X)$ and $\mathbb{B}_\Lambda = \mathbb{B}_{\Lambda,0} \oplus \mathbb{B}_{\Lambda,1}$. Note that $\mathbb{B}_{\Lambda,0} = V_\Lambda$ by definition. Therefore $E_0(X)$ is a direct summand of $V_\Lambda \otimes_{\mathbb{F}_{p^2}} R$. We claim that $\mathrm{rk}_R(E_0(X)) = d+1$ and that $E_0(X)^\perp \subset E_0(X)$. As $E_0(X)$ is direct summand we can check this after base change to an algebraically closed field and then the claim follows from [Vo] 2.12.

We obtain a map $\mathcal{N}_\Lambda(R) \to Y_\Lambda(R)$ by sending (X, ρ_X) to $E_0(X)$. As the formation of $E(X)$ (and hence of $E_0(X)$) commutes with base change, this map is functorial in R and therefore defines the desired morphism f.

Theorem 3.10. *The morphism f is an isomorphism.*

Proof. By [Vo] 2.12 we know that f is a bijection on k-valued points for every perfect extension of $\mathbb{F}_{p^2}$ and hence f is universally bijective. As $\mathcal{N}_\Lambda$ is proper and Y_Λ is separated, f is proper and therefore a universal homeomorphism. In particular we see that $\mathcal{N}_\Lambda$ is geometrically irreducible of dimension $d := t(\Lambda)$.

Moreover, we can use Zink's theory of windows for formal p-divisible groups [Zi] and work systematically with a Cohen ring instead of the Witt ring. Then the arguments of loc. cit. show that f is a bijection on k-valued points for an arbitrary field extension k of $\mathbb{F}_{p^2}$.

Now let k be an algebraically closed extension of $\mathbb{F}_{p^2}$ and let $x \in \mathcal{N}_\Lambda(k)$ be a k-valued point. We show that the tangent space $T_x(\mathcal{N}_\Lambda)$ has dimension

$\leq d$. This proves the theorem. Indeed, as $\mathcal{N}_\Lambda$ is geometrically irreducible of dimension d, $\dim(T_x(\mathcal{N}_\Lambda)) \leq d$ shows that $\mathcal{N}_\Lambda$ is smooth. As we already know that f is finite and birational and that Y_Λ is smooth, Zariski's main theorem shows that f is an isomorphism.

To calculate $T_x(\mathcal{N}_\Lambda)$ we use Grothendieck-Messing theory. The point x corresponds to a unitary p-divisible group (X, ι_X, λ_X) over k endowed with two isogenies

$$X_{\Lambda^-,k} \longrightarrow X \longrightarrow X_{\Lambda^+,k}. \tag{3.7.1}$$

Let $\mathbb{D}^\pm = \mathbb{D}(X_{\Lambda^\pm,k})$ be the crystal of the Lie algebra of the universal vector extension of $X_{\Lambda^\pm,k}$ (see [Me] chap. IV, §2). Similarly set $\mathbb{D} := \mathbb{D}(X)$. We evaluate this crystals at the PD-thickening $\mathrm{Spec}(k) \hookrightarrow \mathrm{Spec}(k[\varepsilon]/(\varepsilon^2))$ where the ideal (ε) is endowed with the trivial PD-structure. We write $\tilde{D}^\pm$ for the evaluation of $\mathbb{D}^\pm$ and similarly $\tilde{D}$ for the evaluation of $\mathbb{D}$. These three modules are free $k[\varepsilon]/(\varepsilon^2)$-modules of rank $2n$ which an O_K-action. Therefore they decompose $\tilde{D}^? = \tilde{D}^?_0 \oplus \tilde{D}^?_1$. Altogether (3.7.1) induces homomorphisms of free modules of rank n

$$\tilde{D}_0^- \xrightarrow{\varphi^-} \tilde{D}_0 \xrightarrow{\varphi^+} \tilde{D}_0^+. \tag{3.7.2}$$

As in the construction of f we see that $\mathrm{Coker}(\varphi^-)$ and $\mathrm{Coker}(\varphi^+)$ are free $k[\varepsilon]/(\varepsilon^2)$-modules of rank $d{+}1$ and rank d, respectively. We identify $\tilde{D}_0^\pm$ with $D(X_{\Lambda^\pm,k[\varepsilon]/(\varepsilon^2)})_0$ and denote by $\tilde{H}_0^\pm \subset D(X_{\Lambda^\pm,k[\varepsilon]/(\varepsilon^2)})_0 = \tilde{D}_0^\pm$ the zeroth component of the Hodge filtration in $D(X_{\Lambda^\pm,k[\varepsilon]/(\varepsilon^2)})$. As the signature of $X_{\Lambda^\pm}$ is $(0,n)$ by definition (3.2), we have $\tilde{H}_0^\pm = \tilde{D}_0^\pm$. Moreover, let $H_0 \subset D(X)_0$ be the zeroth component of the Hodge filtration given by X. As the signature of X equals $(1, n-1)$, H_0 is a subspace of codimension 1.

Now Grothendieck-Messing implies that the tangent space is a subspace of the k-vector space of liftings $\tilde{H}_0$ of H_0 to direct summands of $\tilde{D}_0$ such that $\varphi^-(\tilde{D}_0^-) \subset \tilde{H}_0$. Hence this tangent space can be identified with a subspace of the tangent space of the projective space parametrizing direct summands of corank 1 of the module $\tilde{D}_0/\mathrm{Im}(\varphi^-)$ which is of rank $d+1$. Therefore we have $\dim(T_x(\mathcal{N}_\Lambda)) \leq d$. □

Corollary 3.11. *The closed subscheme $\mathcal{N}_\Lambda$ of $\mathcal{N}_i$ is projective, smooth and geometrically irreducible of dimension $t(\Lambda)$.*

Proof. As Y_Λ has these properties by Lemma 3.5, the corollary follows from Theorem 3.10. □

4 The global structure of $\mathcal{N}$: The Bruhat-Tits stratification

(4.1) Combinatorial structure of $\mathcal{N}$.

We continue to assume that i is an integer such that ni is even, i.e. that $\mathcal{N}_i \neq \emptyset$.

Theorem 4.1. *Let $\Lambda, \tilde{\Lambda} \in \mathcal{L}_i$ be two lattices.*
(1) *Then $\Lambda \subset \tilde{\Lambda}$ if and only if $\mathcal{N}_\Lambda \subset \mathcal{N}_{\tilde{\Lambda}}$. In this case, $t(\Lambda) \leq t(\tilde{\Lambda})$, and $t(\Lambda) = t(\tilde{\Lambda})$ implies $\Lambda = \tilde{\Lambda}$.*
(2) *The following assertions are equivalent.*
 (i) $\Lambda \cap \tilde{\Lambda} \in \mathcal{L}_i$.
 (ii) $\Lambda \cap \tilde{\Lambda}$ *contains a lattice of* $\mathcal{L}_i$.
 (iii) $\mathcal{N}_\Lambda \cap \mathcal{N}_{\tilde{\Lambda}} \neq \emptyset$.
 If these conditions are satisfied,

$$\mathcal{N}_\Lambda \cap \mathcal{N}_{\tilde{\Lambda}} = \mathcal{N}_{\Lambda \cap \tilde{\Lambda}},$$

 where $\mathcal{N}_\Lambda \cap \mathcal{N}_{\tilde{\Lambda}}$ denotes the scheme-theoretic intersection in $\mathcal{N}_i$.
(4) *The following assertions are equivalent.*
 (i) $\Lambda + \tilde{\Lambda} \in \mathcal{L}_i$.
 (ii) $\Lambda + \tilde{\Lambda}$ *is contained in a lattice of* $\mathcal{L}_i$.
 (iii) $\mathcal{N}_\Lambda$ *and* $\mathcal{N}_{\tilde{\Lambda}}$ *are both contained in* $\mathcal{N}_{\Lambda'}$ *for some* $\Lambda' \in \mathcal{L}_i$.
 If these conditions are satisfied, $\mathcal{N}_{\Lambda+\tilde{\Lambda}}$ is the smallest subscheme of the form $\mathcal{N}_{\Lambda'}$ that contains $\mathcal{N}_\Lambda$ and $\mathcal{N}_{\tilde{\Lambda}}$.
(4) *Let k be an algebraically closed extension of $\mathbb{F}_{p^2}$. Then*

$$\mathcal{N}_i(k) = \bigcup_{\Lambda \in \mathcal{L}_i} \mathcal{N}_\Lambda(k).$$

Proof. The nonobvious parts of Assertions (1), (2), and (4) follow from [Vo] Proposition 2.6. Assertion (3) is proved as Assertion (2) by dualizing all lattices. □

Theorem 4.2. *Let $i \in \mathbb{Z}$ be such that ni is even.*
(1) *The scheme $\mathcal{N}_{i,\mathrm{red}}$ is geometrically connected of pure dimension $[\frac{n-1}{2}]$.*
(2) *Sending Λ to $\mathcal{N}_\Lambda$ defines a bijective map*

$$\{\Lambda \in \mathcal{L}_i \text{ of orbit type } [\tfrac{n-1}{2}]\} \leftrightarrow \{\text{irreducible components of } \mathcal{N}_{i,\mathrm{red}}\}.$$

Proof. By Theorem 4.1 (3) the morphism

$$\coprod_{\Lambda \in \mathcal{L}_i} \mathcal{N}_\Lambda \longrightarrow \mathcal{N}_{i,\mathrm{red}} \tag{4.1.1}$$

induced by the inclusions $\mathcal{N}_\Lambda \hookrightarrow \mathcal{N}_i$ is surjective.

The lattices $\tilde{\Lambda} \in \mathcal{L}_i$ with $\Lambda \subsetneq \tilde{\Lambda}$ or $\tilde{\Lambda} \subsetneq \Lambda$ are by definition those vertices in the simplicial complex $\mathcal{B}_i$ which are a neighbour of Λ. Moreover, if $\tilde{\Lambda}$ is such a neighbour of Λ we have $\tilde{\Lambda} \subsetneq \Lambda$ if and only if $t(\tilde{\Lambda}) < t(\Lambda)$ by (3.2.4). In this case we have $\mathcal{N}_{\tilde{\Lambda}} \subset \mathcal{N}_\Lambda$ by Theorem 4.1 (1).

As $\mathcal{B}_i$ is isomorphic to $\mathcal{B}(\tilde{J}, \mathbb{Q}_p)$ it follows from the general theory of Bruhat-Tits buildings that for any two vertices Λ and Λ' of $\mathcal{B}_i$ there exists a sequence of vertices

$$\Lambda = \Lambda_0, \Lambda_1, \dots, \Lambda_N = \Lambda'$$

such that Λ_i and Λ_{i+1} are neighbours and hence either $\mathcal{N}_{\Lambda_i} \subset \mathcal{N}_{\Lambda_{i+1}}$ or $\mathcal{N}_{\Lambda_{i+1}} \subset \mathcal{N}_{\Lambda_i}$. Hence $\mathcal{N}_\Lambda$ and $\mathcal{N}_{\Lambda'}$ are contained in the same connected component of $\mathcal{N}_{i,\mathrm{red}}$. As Λ and Λ' were arbitrary, the surjectivity of (4.1.1) implies that $\mathcal{N}_{i,\mathrm{red}}$ is connected.

For any $\Lambda \in \mathcal{L}_i$ and any integer $\tilde{d}$ with $t(\Lambda) \leq \tilde{d} \leq \frac{n-1}{2}$ there exists a neighbour $\tilde{\Lambda}$ of Λ with $t(\tilde{\Lambda}) = \tilde{d}$ ([Vo] Proposition 2.7). In particular, $\mathcal{N}_\Lambda$ is always contained in some $\mathcal{N}_{\tilde{\Lambda}}$ with $t(\tilde{\Lambda}) = [\frac{n-1}{2}]$. As $\mathcal{N}_{\tilde{\Lambda}}$ is irreducible of dimension $t(\tilde{\Lambda})$, this implies that assertion (2) and also that $\mathcal{N}_{i,\mathrm{red}}$ is of pure dimension $[\frac{n-1}{2}]$. □

For $\Lambda \in \mathcal{L}_i$ we set

$$\begin{aligned} \mathcal{L}_\Lambda &:= \{\, \tilde{\Lambda} \in \mathcal{L}_i \mid \tilde{\Lambda} \subsetneq \Lambda \,\}, \\ \mathcal{N}_\Lambda^0 &:= \mathcal{N}_\Lambda \setminus \bigcup_{\tilde{\Lambda} \in \mathcal{L}_\Lambda} \mathcal{N}_{\tilde{\Lambda}}. \end{aligned} \tag{4.1.2}$$

Proposition 4.3. *The subset $\mathcal{N}_\Lambda^0$ is open and dense in $\mathcal{N}_\Lambda$.*

Proof. The set $\mathcal{L}_\Lambda$ is a finite set, $\mathcal{N}_{\tilde{\Lambda}}$ is a closed subscheme of $\mathcal{N}_\Lambda$ for every $\tilde{\Lambda} \in \mathcal{L}_\Lambda$, and we have

$$\dim(\mathcal{N}_{\tilde{\Lambda}}) = t(\tilde{\Lambda}) < t(\Lambda) = \dim(\mathcal{N}_\Lambda).$$

This implies implies the claim. □

Note that by definition we have a disjoint union of locally closed subschemes

$$\mathcal{N}_\Lambda = \mathcal{N}_\Lambda^0 \uplus \biguplus_{\tilde{\Lambda} \in \mathcal{L}_\Lambda} \mathcal{N}_{\tilde{\Lambda}}^0. \tag{4.1.3}$$

We obtain a locally finite stratification $(\mathcal{N}_\Lambda^0)_{\Lambda \in \mathcal{L}_i}$ of $\mathcal{N}_i$.

Definition 4.4. *The stratification* $(\mathcal{N}^0_\Lambda)_{\Lambda\in\mathcal{L}_i}$ *of* $\mathcal{N}_i$ *is called the* Bruhat-Tits stratification*. The closed subschemes* $\mathcal{N}_\Lambda$ *are called the* closed Bruhat-Tits strata*.*

Theorem 4.1 shows that the intersection behaviour of the closed Bruhat-Tits strata can be read off from the simplicial Bruhat-Tits building $\mathcal{B}(\tilde{J},\mathbb{Q}_p)$. Note that the intersection of two closed Bruhat-Tits strata is always again a closed Bruhat-Tits stratum. In particular it is smooth.

We will now show that the intersection of two $\mathcal{N}_\Lambda$'s can be of any dimension within the bounds given by Theorem 4.1.

Proposition 4.5. *Let d and d' be two integers with $0 \le d, d' \le (n-1)/2$. Let $\Lambda \in \mathcal{L}_i$ be a lattice with $t(\Lambda) = d$.*
(1) *For any integer d_- with $0 \le d_- \le \min\{d, d'\}$ there exists a lattice $\Lambda' \in \mathcal{L}_i$ with $t(\Lambda') = d'$ such that $\mathcal{N}_\Lambda \cap \mathcal{N}_{\Lambda'}$ has dimension d_-.*
(2) *For any integer d_+ with $\max(d, d') \le d_+ \le (n-1)/2$ there exists a lattice $\Lambda' \in \mathcal{L}_i$ with $t(\Lambda') = d'$ such that the smallest subscheme Y of the form $\mathcal{N}_{\tilde{\Lambda}}$ containing $\mathcal{N}_\Lambda$ and $\mathcal{N}_{\Lambda'}$ has dimension d_+.*

Proof. We may assume that $i = 0$. Consider the case that n is even. We can choose a basis $e_1, \dots, e_n$ of $\mathbf{N}_0$ such that $\langle\,,\rangle$ is given by the matrix T_{even} (see (1.6.2)). Let $\Lambda(r_1, \dots, r_n)$ be the lattice generated by $p^{r_j} e_j$, $j = 1, \dots, n$, where $r_j \in \mathbb{Z}$. Then $\Lambda(r_1, \dots, r_n)^\vee = \Lambda(-r_n - 1, -r_{n-1}, \dots, -r_1)$. Therefore $\Lambda(r_1, \dots, r_n) \in \mathcal{L}_0$ if and only if
(a) $r_1 + r_n = 0$
(b) $0 \le r_j + r_{n+1-j} \le 1$ for all $j = 2, \dots, n/2$.
(c) There exists an $j \in \{2, \dots, n/2\}$ such that $r_j + r_{n+1-j} = 0$.
In this case,

$$t(\Lambda(r_1, \dots, r_n)) = \#\{\, 2 \le j \le n/2 \mid r_j + -r_{n+1-j} = 1 \,\}.$$

As $\tilde{J}(\mathbb{Q}_p)$ acts transitively on the set of lattices of type d by Remark 3.1, we may assume that $\Lambda = \Lambda(0^{n/2}, 1^d, 0^{n/2-d})$. If we define

$$\Lambda' := \Lambda(0, 1^{d'-d_-}, 0^{n/2-1-d'+d_-}, 1^{d_-}, 0^{n/2-d_-})$$

we see that $\Lambda \cap \Lambda'$ is a lattice in $\mathcal{L}_0$ of type d_-.

The case that n is odd and the second assertion can be proved similarly. □

In particular, given any irreducible component Y of $\mathcal{N}$ and any integer d with $0 \le d \le (n-1)/2$, there exists an irreducible component Y' such that $\dim(Y \cap Y') = d$.

As a second application we examine how many closed Bruhat-Tits strata lie on a given one. For this we define the following invariants. Let $(W, (\,,\,))$

be a nondegenerate $(\mathbb{F}_{p^2}/\mathbb{F}_p)$-hermitian space of dimension l (unique up to isomorphism). For any $\mathbb{F}_{p^2}$-subspace U let $U^\perp$ the orthogonal space with respect to $(\ ,\ )$. We fix an integer r with $l/2 \leq r \leq l$ and set

$$\begin{aligned} N(r,W) &:= \{\, U \subset V \ \mathbb{F}_{p^2}\text{-subspace} \mid \dim(U) = r, U^\perp \subset U \,\}, \\ \nu(r,l) &:= \#N(r,W). \end{aligned} \tag{4.1.4}$$

We remark that by [Vo] Lemma 2.17, $N(r,W)$ has an interpretation as the set of $\mathbb{F}_{p^2}$-valued points of a Deligne-Lusztig variety of the group $\mathrm{SU}(W)$.

Example 4.6. For all integers $l \geq 2$ we have

$$\nu(l-1,l) = \#X_{l-1}(\mathbb{F}_{p^2}),$$

where $X_l = V_+(X_0^{p+1} + \cdots + X_l^{p+1})$ is a Fermat hypersurface in $\mathbb{P}^l_{\mathbb{F}_{p^2}}$. We claim that

$$\#X_l(\mathbb{F}_{p^2}) = \begin{cases} (p^{l+1}+1)\Sigma_l, & \text{if } l \text{ is even;} \\ (p^l+1)\Sigma_l, & \text{if } l \text{ is odd,} \end{cases}$$

where

$$\Sigma_l := \sum_{j=1}^{[\frac{l-1}{2}]} p^{2j}.$$

In particular

$$\nu(1,2) = p+1, \qquad \nu(2,3) = p^3+1.$$

Let us prove the claim. For this we remark that $\#\mu_{p+1}(\mathbb{F}_{p^2}) = p+1$, where μ_{p+1} is the scheme of $(p+1)$-th roots of unity. Therefore, if $a \in \mathbb{F}_{p^2}^{\times,p+1}$ (i.e. a is a $(p+1)$-th power in $\mathbb{F}_{p^2}^\times$), there are $p+1$ solutions to the equation $X^{p+1} = a$. Moreover, an element $a \in \mathbb{F}_{p^2}^\times$ lies in $\mathbb{F}_{p^2}^{\times,p+1}$ if and only if $a \in \mathbb{F}_p^\times$. We set

$$A_l := \{\, x = (x_1, \dots, x_l) \in \mathbb{A}^l(\mathbb{F}_{p^2}) \mid 1 + x_1^{p+1} + \cdots + x_l^{p+1} = 0 \,\}.$$

If we define $b_l := \#X_l(\mathbb{F}_{p^2})$ and $a_l := \#A_l$, we have $b_l = b_{l-1} + a_l$ and hence

$$b_l = \sum_{j=1}^{l} a_j. \tag{4.1.5}$$

The set $\{\, x \in A_l \mid x_l = 0 \,\}$ has a_{l-1} elements. The complement is the set A_l' of elements x with $x_l^{p+1} = -1 - (x_1^{p+1} + \cdots + x_{l-1}^{p+1}) \neq 0$. For these we can choose $(x_1, \dots, x_{l-1})$ arbitrary as long as $(x_1, \dots, x_{l-1}) \notin A_{l-1}$ and then we have for x_l still $p+1$ possibilities. Therefore we see that $\#A_l' = (p+1)(p^{2(l-1)} - a_{l-1})$ and hence $a_l = (p+1)p^{2(l-1)} - pa_{l-1}$. An easy induction shows that $a_l = p^{2l-1} + (-1)^{l-1}p^{l-1}$ and then the claim follows from (4.1.5) by a second easy induction.

We fix a lattice $\Lambda \in \mathcal{L}_i$ and set $d := t(\Lambda)$. If $\Lambda' \in \mathcal{L}_i$ is a lattice with $\Lambda' \subset \Lambda$ the inclusion induces an $\mathbb{F}_{p^2}$-linear map $\psi_{\Lambda',\Lambda}\colon V_{\Lambda'} \to V_\Lambda$. Now [Vo] 2.10 shows that for a fixed $d' \leq d$ we obtain a bijection

$$\begin{aligned}\{\, \Lambda' \in \mathcal{L}_i \mid \Lambda' \subset \Lambda, t(\Lambda') = d' \,\} &\leftrightarrow N(d + d' + 1, V_\Lambda),\\ \Lambda' &\mapsto \operatorname{Im}(\psi_{\Lambda',\Lambda}).\end{aligned}$$

Note that $l := \dim V_\Lambda = 2d + 1$ and hence $l/2 \leq d + d' + 1 \leq l$. Therefore Theorem 4.1 shows the following corollary (the second part follows by an easy dualizing argument).

Corollary 4.7.
(1) *Let d' be an integer with $0 \leq d' \leq d = t(\Lambda)$. The number of closed Bruhat-Tits strata $\mathcal{N}_{\Lambda'}$ of dimension d' such that $\mathcal{N}_{\Lambda'} \subset \mathcal{N}_\Lambda$ is equal to $\nu(d + d' + 1, 2d + 1)$.*
(2) *Let d' be an integer with $d = t(\Lambda) \leq d' \leq (n-1)/2$. The number of closed Bruhat-Tits strata $\mathcal{N}_{\Lambda'}$ of dimension d' such that $\mathcal{N}_{\Lambda'} \supset \mathcal{N}_\Lambda$ equals $\nu(n - (d + d' + 1), n - (2d + 1))$.*

(4.2) Example $n = 4$.

As any two non-degenerate hermitian spaces of a fixed dimension over a finite field are isomorphic, Y_Λ and hence $\mathcal{N}_\Lambda$ depends up to isomorphism only on the orbit type $t(\Lambda)$.

Example 4.8. If $t(\Lambda) = 0$, $\mathcal{N}_\Lambda$ is a point.

If $t(\Lambda) = 1$, $\mathcal{N}_\Lambda$ is isomorphic to the Fermat curve in $\mathbb{P}^2_{\mathbb{F}_{p^2}}$ given by the equation $x_0^{p+1} + x_1^{p+1} + x_2^{p+1}$ ([Vo] 4.11).

We now apply the general results above to the case $n = 4$. Then for all $i \in \mathbb{Z}$, $\mathcal{N}_i$ is nonempty and the $\mathcal{N}_i$ are the connected components of $\mathcal{N}$. $\mathcal{N}_{i,\mathrm{red}}$ is equi-dimensional of dimension 1. The irreducible components are the closed Bruhat-Tits strata $\mathcal{N}_\Lambda$ of dimension 1. Their nonempty intersections are the closed Bruhat-Tits strata $\mathcal{N}_{\Lambda'}$ of dimension 0.

Each irreducible component $\mathcal{N}_\Lambda$ is isomorphic to

$$V_+(X_0^{p+1} + X_1^{p+1} + X_2^{p+1}) \subset \mathbb{P}^2_{\mathbb{F}_{p^2}}.$$

The intersection points with other irreducible components are precisely the $\mathbb{F}_{p^2}$-rational points of $\mathcal{N}_\Lambda$ and there are $\nu(2,3) = p^3 + 1$ of them. Through each such intersection point go $p^3 + 1$ irreducible components and any two of them intersect transversally.

(4.3) The Ekedahl-Oort strata of $\mathcal{N}_\Lambda$.

Let $i \in \mathbb{Z}$ such that ni is even. The Ekedahl-Oort stratification (2.3) on $\mathcal{N}_i \otimes_{\mathbb{Z}_{p^2}} \mathbb{F}_{p^2}$ induces for every subscheme S a stratification of locally closed subschemes

$$S = \biguplus_{0 \leq \sigma \leq \frac{n-1}{2}} S(\sigma).$$

Theorem 4.9. *Fix an integer σ with $0 \leq \sigma \leq \frac{n-1}{2}$ and let $\Lambda \in \mathcal{L}_i$. Then*

$$\mathcal{N}_\Lambda(\sigma) = \coprod_{\substack{\tilde{\Lambda} \in \mathcal{L}_\Lambda \\ t(\tilde{\Lambda}) = \sigma}} \mathcal{N}^0_{\tilde{\Lambda}}.$$

is a decomposition into open and closed geometrically irreducible subschemes. In particular, $\mathcal{N}_\Lambda(\sigma) = \emptyset$ if and only if $\sigma > t(\Lambda)$.

Proof. By (4.1.3) it suffices to show the following assertion. □

Corollary 4.10. *There exists a decomposition into open and closed geometrically irreducible subschemes*

$$\mathcal{N}_{i,\mathrm{red}}(\sigma) = \coprod_{\substack{\Lambda \in \mathcal{L}_i \\ t(\Lambda) = \sigma}} \mathcal{N}^0_{\Lambda}.$$

Proof. Let k be an algebraically closed extension of $\mathbb{F}_{p^2}$. By Theorem 2.4 we have

$$\mathcal{N}_i(\sigma)(k) = \biguplus_{\substack{\Lambda \in \mathcal{L}_i \\ t(\Lambda) = \sigma}} \mathcal{N}^0_{\Lambda}(k). \tag{4.3.1}$$

Therefore it suffices to show that $\mathcal{N}^0_\Lambda$ is open and closed in $\mathcal{N}_i(\sigma)_{\mathrm{red}}$ for all $\Lambda \in \mathcal{L}_i$ with $t(\Lambda) = \sigma$. By Proposition 4.3, $\mathcal{N}^0_\Lambda$ is open in $\mathcal{N}_\Lambda$. As $\mathcal{N}_\Lambda \cap \mathcal{N}_i(\sigma)_{\mathrm{red}} = \mathcal{N}^0_\Lambda$, the subscheme $\mathcal{N}^0_\Lambda$ is open in $\mathcal{N}_i(\sigma)_{\mathrm{red}}$. Now it follows from (4.3.1) that the complement of $\mathcal{N}^0_\Lambda$ in $\mathcal{N}_i(\sigma)_{\mathrm{red}}$ is also open and therefore $\mathcal{N}^0_\Lambda$ is also closed in $\mathcal{N}_i(\sigma)_{\mathrm{red}}$. □

Theorem 4.11. *The scheme $\mathcal{N}_{i,\mathrm{red}}$ is of pure dimension $[(n-1)/2]$ and locally a complete intersection. The smooth locus of $\mathcal{N}_{i,\mathrm{red}}$ is equal to the Ekedahl-Oort stratum $\mathcal{N}_i([(n-1)/2])$.*

Proof. We set $d := [(n-1)/2]$. We already know that all irreducible components are smooth and of dimension d. Moreover arbitrary intersections of

them are smooth by Theorem 4.1. Let Z and Z' be two irreducible components with $Z \cap Z' \neq \emptyset$ and set $r := \dim(Z \cap Z')$. There exist irreducible components $Z_{d-1}, \dots, Z_{r+1}$ such that $Z \cap Z' \subset Z \cap Z_i$ and $\dim(Z \cap Z_i) = i$ (see Proposition 4.5). This implies that $\mathcal{N}_{i,\mathrm{red}}$ is locally a complete intersection and that its singular locus consists of the points which lie on more than a single irreducible component. Therefore its smooth locus is the union of the $\mathcal{N}_\Lambda^0$ for all $\Lambda \in \mathcal{L}_i$ with $t(\Lambda) = [(n-1)/2]$ by Theorem 4.2. And this union is $\mathcal{N}_i([(n-1)/2])$ by Corollary 4.10. □

We will now describe the Ekedahl-Oort strata $\mathcal{N}_\Lambda(\sigma)$ of $\mathcal{N}_\Lambda$ as Deligne-Lusztig varieties for the group $\tilde{J}_\Lambda$ defined in (3.5). We set $d = t(\Lambda)$ and identify the Weyl system of $\tilde{J}_\Lambda$ with $(S_{2d+1}, \{s_1, \dots, s_{2d}\})$ as in (3.5). As $\mathcal{N}_\Lambda(\sigma) = \emptyset$ for $\sigma > d$ by Theorem 4.9, we assume $\sigma \leq d$. Define

$$I_\sigma := \{s_1, \dots, s_{d-\sigma-1}, s_{2d-\sigma+1}, \dots, s_{2d}\} \subset S$$

and $w_\sigma \in S_{2d+1}$ as the cycle $(d+\sigma+1, d+\sigma, \dots, d+1)$ which sends $d+\sigma+1$ to $d+\sigma$ etc. Clearly $F(I_\sigma) = I_\sigma$ and it is easy to check that $w_\sigma \in {}^{I_\sigma}W^{I_\sigma}$. Note that $I_d = I_{d-1} = \emptyset$, but $w_d \neq w_{d-1}$.

Corollary 4.12. *For all integers $0 \leq \sigma \leq t(\Lambda)$ the isomorphism $\mathcal{N}_\Lambda \xrightarrow{\sim} Y_\Lambda$ induces an isomorphism of $\mathcal{N}_\Lambda(\sigma)$ with the Deligne-Lusztig variety*

$$X_{I_\sigma}(w_\sigma) \otimes_{\mathbb{F}_p} \mathbb{F}_{p^2}.$$

Proof. By Theorem 4.9 this is just a reformulation of [Vo] Corollary 2.24 (note that in loc. cit. the Deligne-Lusztig variety is defined in a slightly different way and hence the $X_{P_\sigma}(w)$ there is the variety $X_{I_\sigma}(w^{-1})$ defined here). □

5 The supersingular locus of the Shimura variety of $\mathrm{GU}(1, n-1)$

(5.1) The Shimura datum of $\mathrm{GU}(1, n-1)$.

Let B be a simple $\mathbb{Q}$-algebra such that $B \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathrm{M}_m(\mathbb{C})$ and such that $B \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \mathrm{M}_m(K)$ where K is the quadratic unramified extension of $\mathbb{Q}_p$ fixed in (1.1). Note that this implies that $\mathbb{K} := \mathrm{Cent}(B)$ is a quadratic imaginary extension of $\mathbb{Q}$ and that p is inert in $\mathbb{K}$. Let * be a positive involution on B, and let $\mathbb{V} \neq 0$ be a finitely generated left B-module. We

fix a symplectic form $\langle\,,\rangle\colon \mathbb{V}\times\mathbb{V}\to\mathbb{Q}$ auch that $\langle bv,v'\rangle = \langle v,b^*v'\rangle$ for all $b\in B$ and $v,v'\in\mathbb{V}$.

We denote by $\mathbb{G}$ the algebraic group over $\mathbb{Q}$ of B-linear symplectic similitudes of $(\mathbb{V},\langle\,,\rangle)$. Then $\mathbb{G}_{\mathbb{R}}$ is isomorphic to the group of unitary similitudes $\mathrm{GU}(r,s)$ of an hermitian space of signature (r,s) for nonnegative integers r and s with $r+s=n:=\dim_{\mathbb{K}}(\mathbb{V})/m$. In particular, $\mathbb{G}$ is a connected reductive group over $\mathbb{Q}$. We denote by E the unique subfield of $\mathbb{C}$ which is isomorphic to $\mathbb{K}$ and let φ_0 and φ_1 be the two isomorphisms $\mathbb{K}\to E$.

By [Ko] Lemma 4.3 there is a unique $\mathbb{G}(\mathbb{R})$-conjugacy class X of homomorphisms $h\colon \mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}_{m,\mathbb{C}})\to\mathbb{G}_{\mathbb{R}}$ such that every $h\in X$ defines a Hodge structure of type $\{(-1,0),(0,-1)\}$ on $\mathbb{V}$ (with the sign convention of [De]) and such that

$$\mathbb{V}_{\mathbb{R}}\times\mathbb{V}_{\mathbb{R}}\to\mathbb{R},\qquad (v,v')\mapsto\langle v,h(\sqrt{-1})v'\rangle$$

is symmetric and positive definite on $\mathbb{V}_{\mathbb{R}}$.

Then $(\mathbb{G},X)$ is a Shimura datum. Its reflex field is either equal to $\mathbb{Q}$ (if $r=s$) or equal to E (if $r\neq s$).

We assume that there exist a *-invariant $\mathbb{Z}_{(p)}$-order O_B of B such that $O_B\otimes\mathbb{Z}_p$ is a maximal order of $B_{\mathbb{Q}_p}$ and an O_B-invariant $\mathbb{Z}_p$-lattice Γ of $\mathbb{V}\otimes_{\mathbb{Q}}\mathbb{Q}_p$ such that the alternating form on Γ induced by $\langle\,,\rangle$ is a perfect $\mathbb{Z}_p$-form. As any maximal order of $M_m(K)$ is conjugate to $M_m(O_K)$, we can and do choose an isomorphism $B\otimes\mathbb{Q}_p\xrightarrow{\sim}M_m(K)$ which identifies $O_B\otimes\mathbb{Z}_p$ with $M_m(O_K)$.

(5.2) The associated moduli space of abelian varieties.

Denote by $\mathbb{A}_f^p$ the ring of finite adeles of $\mathbb{Q}$ with trivial p-th component and fix an open compact subgroup $C^p\subset\mathbb{G}(\mathbb{A}_f^p)$. Let $\mathcal{M}_{C^p}$ be the moduli space of abelian varieties associated with the data $(B,{}^*,\mathbb{V},\langle\,,\rangle,O_B,\Gamma,C^p)$ by Kottwitz [Ko] §5. More precisely, $\mathcal{M}_{C^p}$ is the set-valued functor on the category of $O_{E,(p)}$-schemes which sends each such scheme S to the set of isomorphism classes of tuples $A=(A,\iota_A,\bar\lambda_A,\bar\eta_A)$ where

- A is an abelian scheme over S of relative dimension equal to $\dim_{\mathbb{K}}(\mathbb{V})$.
- $\iota_A\colon O_B\to\mathrm{End}(A)\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}$ is a nonzero (and hence injective) homomorphism of $\mathbb{Z}_{(p)}$-algebras. Note that if $A^\vee$ is the dual abelian scheme of A, we obtain an O_B-action $\iota_{A^\vee}\colon O_B\to\mathrm{End}(A^\vee)\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}$ on $A^\vee$ by setting
$$\iota_{A^\vee}(b):=\iota_A(b^*)^\vee.$$
- $\bar\lambda_A\subset\mathrm{Hom}(A,A^\vee)\otimes_{\mathbb{Z}}\mathbb{Q}$ is a one-dimensional $\mathbb{Q}$-subspace which contains a p-principal O_B-linear polarization.
- $\bar\eta_A$ is a C^p-level structure $\bar\eta_A\colon H_1(A,\mathbb{A}_f^p)\xrightarrow{\sim}V\otimes_{\mathbb{Q}}\mathbb{A}_f^p$ mod C^p.

such that (A, ι_A) satisfies Kottwitz's determinant condition of signature (r, s), i.e. that we have an equality of polynomials

$$\operatorname{charpol}(b, \operatorname{Lie}(A)) = (T - \varphi_0(b))^{mr}(T - \varphi_1(b))^{ms} \in \mathscr{O}_S[T]$$

for all $b \in O_B \cap \mathbb{K}$ (note that $O_B \cap \mathbb{K} = O_{\mathbb{K},(p)}$ and hence $\varphi_i(O_B \cap \mathbb{K}) = O_{E,(p)}$; via the structure morphism $S \to \operatorname{Spec} O_{E,(p)}$ we can therefore consider $\varphi_i(b)$ for $b \in O_B \cap \mathbb{K}$ as sections in $\mathscr{O}_S$).

We call two such tuples $(A, \iota_A, \bar{\lambda}_A, \bar{\eta}_A)$ and $(A', \iota_{A'}, \bar{\lambda}_{A'}, \bar{\eta}_{A'})$ isomorphic, if there exists an O_B-linear isogeny $\psi\colon A \to A'$ of degree prime to p such that $\psi^*(\bar{\lambda}_{A'}) = \bar{\lambda}_A$ and $\bar{\eta}_{A'} \circ H_1(\psi, \mathbb{A}_f^p) = \bar{\eta}_A$.

This functor is represented by a smooth quasi-projective scheme over $O_{E,(p)}$ if C^p is sufficiently small, e.g. if C^p is contained in a principal congruence subgroup of level $N \geq 3$, where N is an integer prime to p. From now on we will assume that this is the case.

We denote by $\mathcal{M}_{C^p}^{\mathrm{ss}}$ the supersingular locus of $\mathcal{M}_{C^p} \otimes \overline{\mathbb{F}}$ considered as a closed reduced subscheme of $\mathcal{M} \otimes \overline{\mathbb{F}}$, where $\overline{\mathbb{F}}$ is an algebraic closure of $\mathbb{F}_{p^2}$.

(5.3) Ekedahl-Oort strata of $\mathcal{M}_{C^p}^{\mathrm{ss}}$.

We identify E_p with $\mathbb{Q}_{p^2}$ (and therefore O_{E_p} with $\mathbb{Z}_{p^2}$ and the residue field of O_{E_p} with $\mathbb{F}_{p^2}$). Let S be an $\mathbb{Z}_{p^2}$-scheme on which p is locally nilpotent. To every S-valued point $A = (A, \iota_A, \bar{\lambda}_A, \bar{\eta}_A) \in \mathcal{M}(S)$ we attach the isomorphism class of a unitary p-divisible group of signature (r, s) over S (in the sense of (1.2)) as follows. Let $X' = A[p^\infty]$ be the p-divisible group of A. As the ring of endomorphisms of X' is p-adically complete, the O_B-action on A induces an action of $O_B \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p = M_m(O_K)$ on X'. Morita equivalence tells us that $X' \mapsto O_K^m \otimes_{M_m(O_K)} X'$ and $X \mapsto O_K^m \otimes_{O_K} X$ are mutually quasi-inverse functors between the category of p-divisible groups X' over S with a left $M_m(O_K)$-action and the category of p-divisible groups with O_K-action over S. We set $X := O_K^m \otimes_{M_m(O_K)} A[p^\infty]$ and denote its O_K-action by ι_X. Choose a p-principal polarization $\lambda_A \in \bar{\lambda}_A$. Then λ_A induces an O_K-linear p-principal polarization λ_X on X. Then (X, ι_X, λ_X) is a unitary p-divisible group over S of signature (r, s). Its isomorphism class is independent of the choice of λ_A.

As in (2.3) we define the *Ekedahl-Oort strata of* $\mathcal{M}_{C^p}^{\mathrm{ss}}$: For this we assume that $r = 1$. Fix an integer $0 \leq \sigma \leq (n-1)/2$. Let $(\mathcal{A}, \iota_{\mathcal{A}}, \bar{\lambda}_{\mathcal{A}}, \bar{\eta}_{\mathcal{A}})$ be the restriction of the universal family over $\mathcal{M}_{C^p}$ to $\mathcal{M}_{C^p}^{\mathrm{ss}}$. Let $\mathcal{X}$ be the attached isomorphism class of a unitary p-divisible group over $\mathcal{M}_{C^p}^{\mathrm{ss}}$. Then we denote by $\mathcal{M}_{C^p}^{\mathrm{ss}}(\sigma)$ the locally closed subscheme of $\mathcal{M}_{C^p}^{\mathrm{ss}}$ such that a morphism $f\colon S \to \mathcal{M}_{C^p}^{\mathrm{ss}}$ of $\mathbb{F}$-schemes factorizes through $\mathcal{M}_{C^p}^{\mathrm{ss}}(\sigma)$ if and only if $f^*\mathcal{X}[p]$ is fppf-locally isomorphic to $(\overline{\mathbb{X}}_\sigma)_S$.

The same arguments as in the proof of Theorem 2.2 show that $\mathcal{M}_{C^p}^{\mathrm{ss}}(\sigma)$ is smooth over $\mathbb{F}$ for all σ.

(5.4) The Rapoport-Zink space and the supersingular locus of the Shimura variety.

Fix a point $\mathbf{A}' = (\mathbf{A}', \iota', \bar{\lambda}', \bar{\eta}') \in \mathcal{M}^{\mathrm{ss}}_{C^p}(\overline{\mathbb{F}})$ and let $\mathbf{X}'$ be the attached unitary p-divisible group over $\overline{\mathbb{F}}$ as in (5.3). As in (1.5) we can define a functor $\mathcal{N}'$ on the category of $\overline{\mathbb{F}}$-schemes S where $\mathcal{N}'(S)$ consists of isomorphism classes of pairs (X, ρ_X) where X is a unitary p-divisible group over S of signature (r, s) and where $\rho_X \colon X \to \mathbf{X}'_S$ is a quasi-isogeny compatible with the additional structures as explained in (1.5).

We now relate $\mathcal{N}$ with $\mathcal{N}'$ and $\mathcal{N}'$ with $\mathcal{M}^{\mathrm{ss}}_{C^p}$. For this we show the following.

Lemma 5.1. *For any two supersingular unitary Dieudonné modules M and M' of signature (r, s) over $\overline{\mathbb{F}}$ there exists an isogeny $M \to M'$ of unitary Dieudonné modules.*

Proof. Let H be any connected unramified reductive group over $\mathbb{Q}_p$. We denote by $B(H)$ the set of σ-conjugacy classes of $H(L)$ where L is the field of fractions of $W(\overline{\mathbb{F}})$. Then $B(H)$ classifies isocrystals with H-structure, we refer to [Ko1] for details.

We consider the hermitian space V introduced in (1.1) as a $\mathbb{Q}_p$-vector space of dimension $2n$ (where $n = r + s$). Let $H := \mathrm{GL}_{\mathbb{Q}_p}(V)$. The standard representation $G \hookrightarrow H$ induces a map $B(G) \to B(H)$ and it suffices to show that this map is injective. Let T be a split maximal torus of H_L and let S be a split maximal torus of G_L such that $S \subset T$. We denote by $X_*(S)$ the abelian group of cocharacters of S and set $\mathscr{N}(G) := (X_*(S) \otimes_{\mathbb{Z}} \mathbb{Q})/W_S$ where W_S is the Weyl group of (G, S). In the same way we define $\mathscr{N}(H)$. Note that $\mathscr{N}(G)$ and $\mathscr{N}(H)$ do not depend (up to unique isomorphism) of the choices of the maximal tori. Kottwitz has defined in [Ko1] a map $\nu_G \colon B(G) \to \mathscr{N}(G)$, the Newton map. For H this is simply the map which associates with an isocrystal of height $\dim(V)$ its Newton polygon. We obtain a commutative diagram

$$\begin{tikzcd} B(G) \arrow[r] \arrow[d, "\nu_G"'] & B(H) \arrow[d, "\nu_H"] \\ \mathscr{N}(G) \arrow[r] & \mathscr{N}(H). \end{tikzcd}$$

As the derived groups of G and H are simply connected, the Newton maps ν_G and ν_H are injective and it suffices to show that $\mathscr{N}(G) \to \mathscr{N}(H)$ is injective.

Now $V \otimes_{\mathbb{Q}_p} L$ is a $K \otimes_{\mathbb{Q}_p} L$-module. As in (1.3.2) we have $K \otimes_{\mathbb{Q}_p} L = L \times L$ and therefore $V_L = V_0 \times V_1$, and the hermitian pairing on V_L restrict to perfect pairing $V_0 \times V_1 \to L$ which we use to identify V_1 with the dual space

V_0^*. Therefore we can identify $G(L)$ with

$$\{\, g = (g_0, g_1) \in \mathrm{GL}(V_0) \times \mathrm{GL}(V_0^*) \mid \exists\, c(g) \in L^\times : g_0^* g_1 = c(g)\,\mathrm{id}_{V_0^*} \,\}.$$

Hence we have

$$\begin{aligned}\mathscr{N}(H) &= \{\, \lambda = (\lambda_1, \dots, \lambda_{2n}) \in \mathbb{Q}^{2n} \mid \lambda_1 \geq \dots \geq \lambda_n \,\},\\ \mathscr{N}(G) &= \{\, \lambda \in \mathscr{N}(H) \mid \exists\, \gamma \in \mathbb{Q} : \lambda_i + \lambda_{2n+1-i} = \gamma \;\forall\, i = 1, \dots, n \,\}\end{aligned}$$

and $\mathscr{N}(G) \to \mathscr{N}(H)$ is simply the inclusion. □

By the lemma there exists a quasi-isogeny $\tau\colon \mathbf{X}_{\overline{\mathbb{F}}} \to \mathbf{X}'$ of unitary p-divisible groups. For any $\overline{\mathbb{F}}$-scheme S we obtain a bijection

$$\mathcal{N}(S) \to \mathcal{N}'(S), \qquad (X, \rho_X) \mapsto (X, \tau \circ \rho_X)$$

which is functorial in S. This defines an isomorphism of formal schemes

$$\zeta\colon \mathcal{N} \otimes_{\mathbb{Z}_{p^2}} \overline{\mathbb{F}} \xrightarrow{\sim} \mathcal{N}'. \tag{5.4.1}$$

The quasi-isogeny τ defines an isomorphism of J with the group of auto-quasi-isogenies J' of $\mathbf{X}'$. If we identify J with J' via this isomorphism, the isomorphism ζ is $J(\mathbb{Q}_p)$-equivariant.

Now $\mathcal{N}'$ and $\mathcal{M}^{\mathrm{ss}}_{C^p}$ are related by the results of [RZ] as explained in § 6 of [Vo]. We recall the main points (note that the situation considered in [Vo] is more special as only the case $n = 3$ and a more special Shimura datum is considered, but the arguments are verbatim the same in this more general case).

Let I be the group of O_B-linear of quasi-isogenies in $\mathrm{End}_{O_B}(\mathbf{A}') \otimes \mathbb{Q}$ which respect the space of polarizations $\bar{\lambda}'$. This is a reductive group over $\mathbb{Q}$, which is an inner form of $\mathbb{G}$. We have isomorphisms of algebraic groups $I_{\mathbb{Q}_p} \xrightarrow{\sim} J$ and $I_{\mathbb{A}_f^p} \xrightarrow{\sim} \mathbb{G}_{\mathbb{A}_f^p}$ where the second one is given by some $\eta' \in \bar{\eta}'$. In this way we consider $I(\mathbb{Q})$ as a subgroup of $J(\mathbb{Q}_p)$ and of $\mathbb{G}(\mathbb{A}_f^p)$.

Denote by $g_1, \dots, g_m \in \mathbb{G}(\mathbb{A}_f^p)$ representatives of the finitely many double cosets in $I(\mathbb{Q})\backslash\mathbb{G}(\mathbb{A}_f^p)/C^p$ and set

$$\Gamma_j = I(\mathbb{Q}) \cap g_j C^p g_j^{-1}$$

for all $j = 1, \dots, m$. Then the subgroup $\Gamma_j \subset J(\mathbb{Q}_p)$ is discrete and cocompact modulo center. By the assumption made on C^p above, Γ_j is torsion free for all j (use Serre's lemma that there is no nontrivial automorphism of an abelian variety which fixes a polarization and the N-division points for $N \geq 3$, $(N, p) = 1$).

The uniformization theorem of Rapoport and Zink now provides us with isomorphisms of schemes over $\overline{\mathbb{F}}$

$$\coprod_{j=1}^{m} \Gamma_j \backslash \mathcal{N}'_{\mathrm{red}} \cong I(\mathbb{Q})\backslash(\mathcal{N}'_{\mathrm{red}} \times \mathbb{G}(\mathbb{A}_f^p)/C^p) \xrightarrow{\sim} \mathcal{M}^{\mathrm{ss}}.$$

Composition with (5.4.1) yields an isomorphism

$$\alpha\colon I(\mathbb{Q})\backslash(\mathcal{N}_{\mathbb{F},\mathrm{red}} \times \mathbb{G}(\mathbb{A}_f^p)/C^p) \xrightarrow{\sim} \mathcal{M}^{\mathrm{ss}}. \tag{5.4.2}$$

Consider the induced surjective morphism

$$\Psi\colon \coprod_{j=1}^{m} \mathcal{N}_{\overline{\mathbb{F}},\mathrm{red}} \xrightarrow{\sim} \coprod_{j=1}^{m} \mathcal{N}'_{\mathrm{red}} \longrightarrow \mathcal{M}^{\mathrm{ss}}. \tag{5.4.3}$$

As explained in the proof of Theorem 6.5 of [Vo], our assumption on C^p (and the consequence that Γ_j is torsion free) implies that the canonical projection

$$\pi_j : \mathcal{N}'_{\mathrm{red}} \to \Gamma_j \backslash \mathcal{N}'_{\mathrm{red}}$$

is a local isomorphism and hence Ψ is a local isomorphism. As Γ_j is discrete, the restriction of π_j to any closed quasi-compact subscheme of $\mathcal{N}'_{\mathrm{red}}$ is finite and therefore the same holds for Ψ.

(5.5) The structure of $\mathcal{M}^{\mathrm{ss}}_{C^p}$.

We now again assume that $r = 1$.

Theorem 5.2. *Let C^p be as above. The supersingular locus $\mathcal{M}^{\mathrm{ss}}_{C^p}$ is of pure dimension $[(n-1)/2]$ and locally of complete intersection. Its smooth locus is the open Ekedahl-Oort stratum $\mathcal{M}^{\mathrm{ss}}_{C^p}([(n-1)/2])$.*

Proof. As the morphism Ψ in (5.4.3) is a local isomorphism, the claim follows from Theorem 4.11 (note that Ψ preserves Ekedahl-Oort strata). □

Fix an integer $0 \leq d \leq (n-1)/2$. Let $i \in \mathbb{Z}$ with ni even. By (3.1) the group $\tilde{J}(\mathbb{Q}_p)$ acts transitively on the set of lattices in $\mathcal{L}_i$ of orbit type d and therefore on the set of closed Bruhat-Tits strata $\mathcal{N}_\Lambda \subset \mathcal{N}_i$ of dimension d. Hence Proposition 1.1 implies that $J(\mathbb{Q}_p)$ acts transitively on the set of all closed Bruhat-Tits strata in $\mathcal{N}$ of dimension d. Choose some $\Lambda^{(d)}$ in, say, $\mathcal{L}_0$ such that $t(\Lambda^{(d)}) = d$ and let $C_p^{(d)}$ be its stabilizer in $J(\mathbb{Q}_p)$. Then we obtain a bijection

$$J(\mathbb{Q}_p)/C_p^{(d)} \leftrightarrow \{\text{closed Bruhat-Tits strata in } \mathcal{N} \text{ of dimension } d\}. \tag{5.5.1}$$

By Corollary 3.11 the closed Bruhat-Tits strata $\mathcal{N}_\Lambda$ of $\mathcal{N}$ are smooth, geometrically irreducible, and projective. For some $j = 1, \dots, m$, we consider $\mathcal{N}_\Lambda \otimes \mathbb{F}$ as a subscheme in the j-th copy of $\mathcal{N}_{\mathrm{red}} \otimes \mathbb{F}$ in the left hand side of (5.4.3). The restriction of Ψ to this copy is finite. Its image $\mathcal{M}_{\Lambda,j}$ under Ψ is a (geometrically) irreducible and projective subscheme of $\mathcal{M}^{\mathrm{ss}}_{C^p}$ of dimension $t(\Lambda)$. We call the subschemes of $\mathcal{M}^{\mathrm{ss}}_{C^p}$ of this form the *closed Bruhat-Tits subschemes of* $\mathcal{M}^{\mathrm{ss}}_{C^p}$.

Note that the closed Bruhat-Tits subschemes of $\mathcal{M}^{\mathrm{ss}}_{C^p}$ of dimension $[(n-1)/2]$ are the irreducible components of $\mathcal{M}^{\mathrm{ss}}_{C^p}$ and that the closed Bruhat-Tits subschemes of dimension 0 are the superspecial points of $\mathcal{M}^{\mathrm{ss}}_{C^p}$. By (4.3) the closed Bruhat-Tits subschemes of dimension d are the irreducible components of the closure of the Ekedahl-Oort stratum $\mathcal{M}^{\mathrm{ss}}_{C^p}(d)$ (5.3). The bijections (5.4.2) and (5.5.1) therefore show:

Proposition 5.3. *For all integers $0 \leq d \leq (n-1)/2$ there are bijections of finite sets*

$$\left\{\begin{matrix}\textit{irreducible components}\\ \textit{of } \overline{\mathcal{M}^{\mathrm{ss}}_{C^p}(d)}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\textit{closed Bruhat-Tits subschemes}\\ \textit{of } \mathcal{M}^{\mathrm{ss}}_{C^p} \textit{ of dimension } d\end{matrix}\right\}$$
$$\leftrightarrow I(\mathbb{Q})\backslash(J(\mathbb{Q}_p)/C_p^{(d)} \times \mathbb{G}(\mathbb{A}_f^p)/C^p).$$

Note that $\overline{\mathcal{M}^{\mathrm{ss}}_{C^p}([(n-1)/2])} = \mathcal{M}^{\mathrm{ss}}_{C^p}$. In particular we get an expression for the number of irreducible components of $\mathcal{M}^{\mathrm{ss}}_{C^p}$.

The stabilizer of the connected component $\mathcal{N}_0$ of $\mathcal{N}$ in $J(\mathbb{Q}_p)$ is

$$J^0 := \{\, g \in J(\mathbb{Q}_p) \mid v_p(c(g)) = 0 \,\},$$

where $c(g)$ is the multiplier of the unitary similitude g and v_p denotes the p-adic valuation. As the $\mathcal{N}_i$ are connected by Theorem 4.2, we obtain a bijection

$$\mathbb{Z} = J(\mathbb{Q}_p)/J^0 \leftrightarrow \{\text{connected components of } \mathcal{N}\}. \tag{5.5.2}$$

and therefore the following proposition (where we use that by Theorem 4.2 the connected components of $\mathcal{N}$ are geometrically connected).

Proposition 5.4. *There is a bijection of finite sets*

$$\{\textit{connected components of } \mathcal{M}^{\mathrm{ss}}_{C^p}\} \leftrightarrow I(\mathbb{Q})\backslash(J(\mathbb{Q}_p)/J^0 \times \mathbb{G}(\mathbb{A}_f^p)/C^p).$$

The groups $C_p^{(d)}$ are contained in J^0 for all d. They are special parahoric subgroups of J corresponding to vertices in the Bruhat-Tits building. Via the Propositions 5.4 and 5.3 the fibres of the canonical surjection

$$I(\mathbb{Q})\backslash(J(\mathbb{Q}_p)/C_p^{(d)} \times \mathbb{G}(\mathbb{A}_f^p)/C^p) \to I(\mathbb{Q})\backslash(J(\mathbb{Q}_p)/J^0 \times \mathbb{G}(\mathbb{A}_f^p)/C^p)$$

are in bijection with the set of irreducible components of the Ekedahl-Oort strata of dimension d within a given connected component of $\mathcal{M}^{\mathrm{ss}}_{C^p}$.

It can be checked that the number of elements in each of these fibres becomes arbitrarily large if p goes to infinity. In particular we see that the closed Ekedahl-Oort strata of $\mathcal{M}^{\mathrm{ss}}_{C^p}$ are highly reducible for large p.

Recall from (3.5) that we defined for each $\Lambda \in \mathcal{L}_i$ (i an integer with ni even) a Deligne-Lusztig variety Y_Λ whose isomorphism class depended only on $d = t(\Lambda)$. We set $Y_d := Y_\Lambda$ (see Example 4.8 for the shape of Y_0 and Y_1).

Proposition 5.5. *Fix an integer $0 \leq d \leq (n-1)/2$. For sufficiently small C^p, all closed Bruhat-Tits subschemes of $\mathcal{M}^{\mathrm{ss}}_{C^p}$ of dimension d are isomorphic to Y_d and therefore smooth. In particular, all irreducible components of $\mathcal{M}^{\mathrm{ss}}_{C^p}$ are isomorphic to $Y_{[\frac{n-1}{2}]}$ (and therefore smooth).*

Proof. For sufficiently small C^p, the restriction of Ψ to one (or, equivalently, to every) irreducible component $\mathcal{N}_{\tilde{\Lambda}}$ (with $t(\tilde{\Lambda}) = [(n-1)/2]$) is an isomorphism onto its image by [Vo] 6.5. This implies that the same is true for all closed Bruhat-Tits strata $\mathcal{N}_\Lambda$ of $\mathcal{N}$ because all of them are contained in some irreducible component. Therefore Ψ induces an isomorphism of $\mathcal{N}_\Lambda \cong Y_d$ (with $d = t(\Lambda)$) with its image. □

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